Consequences of scaling in nonlinear partial differential equations*

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Scaling of the form $x = ax'$, $t = a^h t'$, $q = a^m Q$ of a nonlinear partial differential equation for q is connected with the form of the auxiliary functions in an inverse scattering method (AKNS scheme). Solvability of an equation by this scheme is treated. It is shown that only equations with $m = 2$ are solvable by using the method of inverse scattering in conjunction with the Schrödinger eigenvalue equation. The criterion $m = 2$ restricts the form of the terms in these equations. The terms, powers of q and its derivatives, can be found by inspection. A separate problem, the decay of a single soliton in the Korteweg–de Vries equation with damping, is solved using only scaling properties.

1. INTRODUCTION

Many nonlinear partial differential equations are invariant under scaling. An example is the Korteweg–de Vries equation[1] (KdV)

$$q_t + 12qq_x + q_{xxx} = 0. \tag{1.1}$$

The scaling

$$x' = ax, \quad t' = a^h t, \quad q(x,t) = a^m Q(x', t'), \tag{1.2}$$

with the choice

$$h = 3, \quad m = 2,$$

leaves (1.1) invariant. The scaling has been used to find polynomial conservation laws.[2,3]

Here we will give some other consequences of this scaling invariance, and give some examples. In Sec. 2 scaling in the formulation of Ablowitz *et al.*[4] (AKNS scheme for short) is treated. Assuming a scalable equation, it is shown that only terms of a certain type can occur in the various auxiliary polynomials. It should be stated here that some nonscalable equations are exactly solvable in the AKNS scheme. Such equations are not treated. A restriction is found for solvability of a scalable equation. In Sec. 3 we prove that a scaling invariant equation can only be solved by an inverse scattering method on the Schrödinger equation when $m = 2$ in (1.2). This determines the type of terms in that equation. It is expected that possible higher order inverse scattering schemes have similar properties. Nonlinear partial differential equations often describe some physical situation to a first order approximation in some small parameter. Dissipation of energy (damping) can be an important higher order correction to the equation. As an example, the KdV equation with damping is treated in Sec. 4. The functional form of the time dependence of a single soliton is shown to depend only on the power in the damping law. Only the scaling properties are used in the derivation.

2. SCALING IN THE AKNS SCHEME

Some nonlinear partial differential equations can be solved exactly by an inverse scattering method with the set of linear equations[4]

$$v_{1x} + i\zeta v_1 = qv_2, \quad v_{2x} - i\zeta v_2 = rv_1, \tag{2.1}$$

where

$$v_{1t} = Av_1 + Bv_2, \quad v_{2t} = Cv_1 - Av_2. \tag{2.2}$$

For equations invariant to a shift in x and t the functions A, B, C, and r are functions of x and t as functionals of $q, q_x, \cdots$. They can also be functions of the parameter ζ. The nonlinear equation of interest is one of the consistency conditions of (2.1) and (2.2), obtained by cross differentiation

$$q_t - 2Aq - B_x - 2i\zeta B = 0. \tag{2.3}$$

Two other conditions are

$$A_x = qC - rB, \tag{2.4a}$$

$$C_x = r_t + 2Ar + 2i\zeta C. \tag{2.4b}$$

For A, B, C, and r we choose polynomials in q, q_x, . . . , and ζ. In addition, the equation for r, (2.4b), has to be consistent with (2.3).

The scaling (1.2) leads to restrictions on the type of terms we can have in these polynomials. Without loss of generality we can take for the scaling of the new quantities in (2.1),

$$v_1 = a^j V_1(x', t'), \quad v_2 = V_2(x', t'),$$
$$\zeta = aZ, \quad r = a^n R(x', t'). \tag{2.5}$$

Equation (2.1) is invariant if we choose for j and n

$$j = m - 1, \quad n = 2 - m. \tag{2.6}$$

Equation (2.2) is also invariant, and A, B, and C scale as

$$A = a^h A'(Q(x', t'), Q'_x, \ldots, Z),$$
$$B = a^{h+j} B'(Q', Q'_x, \ldots, Z), \tag{2.7}$$
$$C = a^{h-j} C'(Q', Q'_x, \ldots, Z).$$

A, B, and C are homogeneous functions of a. Often A, B, and C are polynomials in q, the derivatives q_x, q_{xx}, . . . , and ζ. In this case each term in the polynomial must have the same scaling power as the polynomial itself. This determines the type of terms that can occur in this polynomial. The terms in polymial conservation laws can be found this way.

As an example, consider the KdV equation, for which $m = 2$ and $h = 3$. It follows that $j = 1$,

$$A = a_1\zeta^3 + a_2\zeta q + a_3 q_x,$$
$$B = b_1\zeta^4 + b_2\zeta^2 q + b_3\zeta q_x + b_4 q_{xx} + b_5 q^2, \tag{2.8}$$
$$C = c_1\zeta^2 + c_2 q,$$

and r is constant, as its power $n=0$. These are indeed the terms that appear. The complex coefficients $a_1, \ldots, c_2$ can easily be calculated.

For the nonlinear Schrödinger equation [Eq. (2.14) with $s=2$], $m=1$ and $h=2$. We find for A

$$A = a_1\zeta^2 + a_2\zeta q + a_3\zeta q^* + a_4 qq^* + a_5 q_x + a_6 q_x^*. \tag{2.9}$$

B and C contain the same terms, as the parameter $j=0$. For r, with parameter $n=1$, we have

$$r = d_1\zeta + d_2 q + d_3 q^*. \tag{2.10}$$

Not all terms actually appear, as (2.3) and (2.4) give additional restrictions. The MKdV equation [Eq. (2.12) with $s=2$] also has the forms (2.9) and (2.10).

The different terms in the equation for $u = v_1/v_2$, can be obtained. However, not all terms have the same invariance properties under the transformation $(u,\zeta) \to (\pm u, \pm\zeta^*)$ used in the derivation of Bäcklund transformations,[5] so the Bäcklund transformation cannot be found by inspection. If it exists, it has the right scaling properties (compare the explicit forms in Ref. 5). This scaling of the Bäcklund transformation was known earlier for the sin-Gordon equation.[6]

The homogeneity of A, B, and C has another consequence. Successive partial differentiations of for example A, with respect to the continuous parameter a (Q, x', t' and Z fixed) at $a=1$ (Euler's theorem) gives

$$\zeta\frac{\partial A}{\partial\zeta} + mq\frac{\partial A}{\partial q} + (m+1)q_x\frac{\partial A}{\partial q_x} + \cdots = hA,$$
$$\zeta^2\frac{\partial^2 A}{\partial\zeta^2} + 2m\zeta q\frac{\partial^2 A}{\partial\zeta\partial q} + m^2q^2\frac{\partial^2 A}{\partial q^2} + 2m(m+1)qq_x\frac{\partial^2 A}{\partial q\partial q_x} + \cdots = h(h-1)A, \tag{2.11}$$

For h positive and integral the rhs is zero after h differentiations. For positive integer m this can only happen if each of the partial derivatives is zero. Thus A, B, and C are polynomials in $q, q_x, \ldots$, and ζ. If $m=0$, all terms except the ones with q have to be zero. A can then be an arbitrary function of q, times a function of $q_x, \ldots$, and ζ. This is the case for the sin-Gordon equation.[4,7] For $m=-1$ we have the same property with q_x.

The above can decide whether a given equation fits the AKNS scheme. For example, for a generalized KdV equation

$$q_t + 2(s+1)(s+2)/s^2 q^s q_x + q_{xxx} = 0, \tag{2.12}$$

with $s=\frac{1}{2}$,[8] the parameter $m=4$. With (2.6) we have $j=3$ and $n=-2$. B is of the sixth order in a, and has to contain a term $q^{3/2}$. This gives the term $q^{1/2}q_x$ in (2.12). But then the sixth partial derivative with respect to q does not give zero, (2.11) is violated, and we conclude that (2.12) does not fit in the AKNS scheme.

If the order s in (2.12) is greater than 2, we have the noninteger scaling

$$ms = h - 1 = 2, \quad m = 2/s. \tag{2.13}$$

Similarly, the equations with $s>2$ do not fit the AKNS scheme. B is of order $2+2/s$ in q, and contains a term q^{s+1}. The rhs has noninteger power, and does not vanish after $s+1$ differentiations. This excludes a polynomial for B. The same is true for higher order nonlinear Schrödinger equations

$$iq_t + q_{xx} + 2(s+2)/s^2|q|^s q = 0. \tag{2.14}$$

Here A has a term $|q|^s$, giving $|q|^s q$ in the equation. After h differentiations the lhs still contains a term q^{s-h}, and (2.11) is violated.

Another way to decide whether a given equation fits the AKNS scheme uses the conservation laws. An equation that fits the AKNS scheme has an infinite number of these.[7] It is easily verified that there is one for each scaling power,

$$I_n(q) = a^n I_n'(Q). \tag{2.15}$$

Some way be trivial, for example, the even ones for the KdV equation.

Conversely, an equation that does not have the sequence (2.15) does not fit in the AKNS scheme. It is relatively straightforward to find a nontrivial conservation law with a given scaling, if it exists. By verifying the first few conservation laws we can see easily if a given equation fits the AKNS scheme. As an example, for the generalized KdV equation (2.12) we find conservation laws of power $2/s-1$ and $4/s-1$,

$$\frac{\partial}{\partial t} I_{2/s-1} = \frac{\partial}{\partial t}\int q\,dx = 0, \tag{2.15a}$$

$$\frac{\partial}{\partial t} I_{4/s-1} = \frac{\partial}{\partial t}\int q^2\,dx = 0, \tag{2.15b}$$

but a conservation law with power 1,

$$\frac{\partial}{\partial t}\int q^s\,dx = -\frac{s(s-1)(s-2)}{2}\int q^{s-3}q_x^{\,3}\,dx, \tag{2.15c}$$

only exists for $s=1$ and $s=2$. As seen above, (2.12) for $s=1/2$ or $s>2$ does not fit the AKNS scheme.

3. EQUATIONS SOLVABLE BY THE INVERSE SCATTERING METHOD FOR THE SCHRÖDINGER EQUATION

From (2.1) we have

$$v_{2xx} + (\zeta^2 - rq)v_2 + r_x v_1 = 0. \tag{3.1}$$

This is the Schrödinger equation for an inverse scattering method, with rq as potential, if r_x is zero.

Then the parameter n is zero, and we have with (2.6)

$$n=0, \quad m=2, \quad j=1. \tag{3.2}$$

So only equations with scaling parameter $m=2$ can be solved by inverse scattering on the Schrödinger equation. The parameter h is not determined by this argument. The terms in the equation for q are now completely determined for given h. As the first example, for $h=3$ we find the terms of the KdV equation (1.1).

For $h=5$ we have the form

$$q_t + P_1 q^2 q_x + (P_2 q q_{xxx} + P_3 q_x q_{xx}) + q_{xxxxx} = 0, \tag{3.3}$$

where terms transformable into one another have been grouped together with parentheses. We can establish the terms in A, B, and C, as done above, and calculate the constants p with (2.3) and (2.4). This gives

$$P_1 = 3rP_2, \quad P_3 = 2P_2. \tag{3.4}$$

Equation (3.3) is identical to the one given by Gardner *et al.*[1] (p. 132), after an independent scaling of q and t to change the coefficients. The next one, for $h=7$, is obtained in a similar way. With $r=-1$ we find

$$q_t + 110q^3q_x + 70(q^2q_{xxx} + 4qq_xq_{xx} + q_x{}^3) + 14(qq_{\mathrm{V}} + 3q_xq_{\mathrm{IV}} + 5q_{xx}q_{xxx}) + q_{\mathrm{VII}} = 0. \tag{3.5}$$

(The roman subscripts denote the number of x derivatives.) Existence of these equations is not guaranteed by scaling alone, as the recursion relations for the coefficients of the terms in A, B, and C which follow from (2.3) and (2.4) are overdetermined.

Another way of calculating the coefficients in (3.3) is with the conservation laws. The terms in the conservation laws only depend on m, but the coefficients can differ with the different equations. As we want (3.3) to be exactly solvable we can assume the same conservation laws as for the KdV equation. From (2.15b) we find the last ratio of (3.4), and from the next conservation law

$$I_5 = \int (q^3 + bq_x^2)\,dx, \qquad P_1 = -(3/2b)P_2, \quad P_2 = -15/3b. \tag{3.6}$$

With the coefficient $b = -\frac{1}{2}$ as for the KdV equation we have the first ratio of (3.4), with $r=-1$ and $P_2 = 10$. The coefficients in (3.5) could be found similarly. We then have to use I_7 also.

4. DAMPING IN THE KdV EQUATION

The KdV equation with damping will be treated in this section, as a different application of the scaling properties. Energy dissipation, accounted for by damping, is an important higher order effect in some physical applications of the KdV equation.[10] If the damping is weak it can be taken into account by damping each Fourier mode separately.[11] Our example, the KdV equation with damping, is

$$q_t + 12qq_x + q_{xxx} + \mathrm{FT}^{-1}[\gamma(k)q(k)] = 0. \tag{4.1}$$

FT denotes Fourier transformation, $\gamma(k)$ is the damping, and $q(k)$ is the Fourier transform of $q(x,t)$. A stationary soliton solution of the equation without damping will now decay slowly. We will show, by a simple scaling argument, that the time dependence of the decay only depends on the power d in the damping law

$$\gamma(k) = \epsilon|k|^d, \quad \epsilon \ll 1. \tag{4.2}$$

The soliton balances nonlinearity and dispersion. As these are the dominant effects, we expect the soliton to keep its functional form

$$Q = \frac{1}{\cosh^2(x-4t)} + O(\epsilon). \tag{4.3}$$

We assume[10,11] that the weak damping slowly changes the scale a, $a=a(t)$, of the scaling (1.2). The decay of the soliton can then be found by[12] the equation for the energy of the soliton part,

$$\frac{\partial}{\partial t}\int q^2\,dx + 4\pi\int \gamma_k(k)|q(k)|^2\,dk = 0. \tag{4.4}$$

The calculations are now simple. For the Fourier transform $q(k)$ of q we have the scaling, with (1.2) and $m=2$

$$q(k) = a^{m-1}Q(k') = \frac{1}{2\pi}\int q(x)\exp(-ikx)\,dx, \tag{4.5}$$

$$k' = k/a.$$

Then an equation for $a(t)$ follows, using the scaling (1.2) and (4.5),

$$\frac{\partial a}{\partial t} = -\frac{B\epsilon}{A(2m-1)}a^{d+1}, \tag{4.6}$$

where A and B are the constants

$$A = \int Q^2\,dx' = 4\pi\int_0^\infty |Q(k')|^2\,dk',$$

$$B = 4\pi\int_0^\infty |k'|^d|Q(k')|^2\,dk'. \tag{4.7}$$

The solution of (4.6) is

$$a = \frac{a(0)}{(1+\epsilon\nu t)^{1/d}}, \qquad \nu = \frac{a^{d-1}(0)}{(2m-1)}\frac{B}{A}d, \tag{4.8}$$

together with the limiting case of exponential decay when $d=0$. The manner of decay is only determined by the power d in the damping law, and not by the various parameters in the undamped equation, or by the initial condition. These enter only in the constants.

The constant B/A can be calculated by taking the Fourier transform of (4.3). It follows that

$$\frac{B}{A} = \frac{\Gamma(d+3)\zeta(d+2)}{\Gamma(3)\zeta(2)\pi^d} \tag{4.9}$$

where ζ is the Riemann zeta function. The result (4.8), with the constant (4.9) and $m=2$ contains the four cases treated before.[11]

The calculation proceeds in the same way for other nonlinear equations with damping (or growth) terms. Examples are the generalized KdV equation (2.12), for $s \neq 4$, and the nonlinear Schrödinger equation.

CONCLUSIONS

Some implications of the scaling (1.2) have been treated. Scaling both explains the form of the various auxiliary functions in the AKNS scheme, and yields the various equations solvable by the inverse scattering method using the Schrödinger equation. Possibilities for their terms can be found by a simple inspection. Solvability of an equation in the AKNS scheme is restricted by the scaling properties. In a separate application of the scaling properties we have shown that the time dependence of the decay of a single soliton in the KdV equation follows from scaling only.

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