Scaling invariance of helical curve motion and soliton equations[a)]

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(Received 5 May 1977)

The scaling properties of the equations describing the motion of helical curves determine the scaling of the associated nonlinear evolution equations. Only two polynomial scaling-invariant evolution equations can be found. Of these, the nonlinear Schrödinger equation has the physically correct scaling invariance, but the modified Korteweg–de Vries can not be connected to realistic helical curves.

Many physical phenomena and the equations that describe them are invariant under a change in scale. Scaling invariance of a linear equation leaves the dependent variable unaffected, but in nonlinear equations the dependent variable must typically scale in some specific way to retain the invariance, depending on the equation.

In an extension of Hasimoto's work,[1] Lamb[2] recently found an interesting connection between various nonlinear evolution equations and the motion of vortex filaments or helical curves. The vortex motion equations are linear, and yield in a natural way the linear inverse scattering equations associated with the nonlinear evolution equations. The equations in question are the sine–Gordon and Hirota[3] equations. The latter contains the modified Korteweg–de Vries and nonlinear Schrödinger equation as special cases.

This paper shows that scaling invariance[4] of the vortex equations (1) is consistent only with one particular scaling of the nonlinear evolution equation, namely that of the nonlinear Schrödinger equation. Specifically, only this equation is connected to physically realizable vortex motion, i.e., motion with a given scaling invariant circulation. The other evolution equations discussed by Lamb can be obtained by allowing the circulation to change with scaling, or by considering a different dependent variable.

It is worth noting that invariance transformations of nonlinear evolution equations have been investigated recently in some detail.[5] That work exploited group-theoretical properties of infinitesimal invariance transformations. The scaling transformation used here is finite, and can be generated by iteration of its infinitesimal counterpart. We consider the finite scaling transformation because it is a fairly obvious and convenient, but yet nontrivial and hence an attractive means for a preliminary investigation of nonlinear equations. For completeness we note that nonlinear equations with soliton behavior need not be scaling invariant.[3,4]

The association of nonlinear equations with helical motion proceeds as follows.[1,2]

The Serret–Frenet equations are

$$\hat{t}_s = \kappa\hat{n}, \tag{1a}$$

$$\hat{b}_s = -\tau\hat{n} \tag{1b}$$

$$\hat{n}_s = \tau\hat{b} - \kappa\hat{t}, \tag{1c}$$

where the subscript denotes partial differentiation with respect to the arc length s and the functions $\kappa(s,t)$ and $\tau(s,t)$ are curvature and torsion respectively, which also depend on the time t. The tangent vector $\hat{t}$ is defined by the derivative of the position vector $X(s,t)$,

$$\hat{t} \equiv \mathbf{X}_s(s,t), \tag{1d}$$

while $\hat{n}$ and $\hat{b}$ are the normal and binormal to the curve.

The motion of the vortex is approximated by

$$\mathbf{X}_t = G\kappa\hat{b}, \tag{1e}$$

where G is proportional to the circulation, the integral of fluid velocity around the vortex. The vortex strength G is constant for any one vortex, and can be chosen unity by suitable normalization of time t.

With introduction of the complex vector $\mathbf{N}(s,t)$,

$$\mathbf{N} \equiv (\hat{n} + i\hat{b})\exp[i\int_{-\infty}^{s} ds'(\tau - \tau_0)], \tag{2a}$$

and the complex scalar

$$\psi \equiv \kappa\exp[i\int_{-\infty}^{s} ds'(\tau - \tau_0)], \tag{2b}$$

(τ_0 is the asymptotic value of the torsion as $|s| \to \infty$), combination of Eqs. (1a)–(1c) yields

$$\mathbf{N}_s + i\tau_0\mathbf{N} = -\psi\hat{t}, \tag{3a}$$

$$\hat{t}_s = \tfrac{1}{2}(\psi^*\mathbf{N} + \psi\mathbf{N}^*). \tag{3b}$$

The function ψ will be the dependent variable in the nonlinear evolution equation, and is assumed to vanish as $|s| \to \infty$.

The norm-preserving variation of $\mathbf{N}$ and $\hat{t}$ in time, on the other hand, can be written as[1]

$$\mathbf{N}_t = iR\mathbf{N} + \gamma\hat{t}, \tag{4a}$$

$$\hat{t}_t = -\tfrac{1}{2}(\gamma^*\mathbf{N} + \gamma\mathbf{N}^*), \tag{4b}$$

where $R(s,t)$ is real and $\gamma(s,t)$ complex. The equation of motion (1e) can be expressed as

$$\mathbf{X}_t \equiv C^*\psi^*\mathbf{N} + C\psi\mathbf{N}^* + \theta\hat{t}, \quad C \equiv \tfrac{1}{2}(\zeta + i\eta), \tag{4c}$$

where ζ, η, and θ are real functions of s and t yet to be determined. Equating mixed second derivatives of

[a)]Work done under the auspices of the U.S. Energy Research and Development Administration.

 0022-2488/78/1904-0898$1.00

N from (3) and (4) yields

$$\psi_t + \gamma_s + i(\tau_0\gamma - R\psi) = 0, \tag{5a}$$

$$R_s = \tfrac{1}{2}i(\gamma\psi^* - \gamma^*\psi). \tag{5b}$$

Furthermore, use of $\mathbf{X}_{st} = \mathbf{X}_{ts}$ gives[2]

$$-\tfrac{1}{2}\gamma = (C\psi)_s + i\tau_0 C\psi + \tfrac{1}{2}\theta\psi, \tag{6a}$$

$$\theta_s = \zeta\,|\psi|^2, \tag{6b}$$

and, using Eq. (2b),

$$R_s = (\eta\,|\psi|^2)_s - \tfrac{1}{2}\eta\,|\psi|^2{}_s - \theta_s\tau. \tag{6c}$$

The desired evolution equation for ψ, Eq. (5a), contains only one time derivative. The auxiliary functions R, γ, ζ, and θ are related by Eqs. (5b)–(6c). The linear inverse scattering equations, which allow us to solve Eq. (5a) for ψ analytically, follow from (3) and (6) as shown by Lamb. They contain R and γ with τ_0 as the eigenvalue.

How do Eqs. (1)–(6) behave under a scaling transformation of the spatial coordinate s and time t? This transformation has the form

$$s' = \alpha s, \tag{7a}$$

$$t' = \alpha^h t. \tag{7b}$$

The scaling variable α and exponent h are real.

We first consider scaling of the spatial coordinate s. The spatial position X should transform like s, Eq. (7a). Thus the tangent vector $\hat{t}$, Eq. (1d), is invariant, consistent with the physical meaning of $\hat{t}$ as the unit vector tangent to curve X(s). The scaling of curvature $\kappa(s,t)$, torsion $\tau(s,t)$, and thus the dependent variable ψ is, from Eqs. (1a)–(1c), given by

$$\psi(s,t) = \alpha\psi'(s',t'). \tag{7c}$$

The phase of ψ,

$$\sigma(s,\tau) \equiv \int_{-\infty}^{s} ds'(\tau - \tau_0), \tag{7d}$$

is invariant: Only the magnitude of ψ changes under scaling. The unit vector $\hat{n}$, $\hat{b}$, and thus N, are invariant as they should.

At this point we can already negate a direct association between helical curve motion and those nonlinear evolution equations with a scaling different from Eq. (7c). An example is: the Korteweg–de Vries equation $\phi_t + \phi\phi_s + \phi_{sss} = 0$. Comparing the terms $\phi\phi_s$ and ϕ_{sss} it is clear that ϕ scales according to $\phi(s) = \alpha^2\phi'(s')$, in contrast to Eq. (7c). Thus the Korteweg–de Vries equation can not be identified with the evolution equation (5). [We refrain from additional transformations on the dependent variables, such as the Miura transformation,[6] which connects the KdV equation to the modified KdV equation: The MKdV equation, with nonlinear term $\phi^2\phi_s$, has the correct scaling (7c).]

Having studied the scaling properties of the purely geometrical part of helical curve motion, we now proceed to examine scaling of time according to Eq. (7b) in the equation of motion (1e). The parameter G, proportional to an integral of velocity x length, scales as $G = \alpha^{2-h}G$. When in addition we use the scaling of κ, we see that Eq. (1e) is only scaling invariant when $h = 2$. Hasimoto[1] found the corresponding nonlinear Schrödinger equation, which is the only possible one as will become clear later.

However, we could artificially consider a vortex with circulation G dependent on the scaling parameter α as $G(\alpha) = G(\alpha = 1)\alpha^{2h-4}$. Equivalently, we could choose Eq. (1e) for our dynamical equation, but refrain from an interpretation in terms of vortices. Then nothing compels us to take $h = 2$, and we can consider h an arbitrary real constant.

With this assumption it follows from the defining relations (4) that the auxiliary functions R and γ, η and ζ, and θ scale as

$$R(s,t) = \alpha^h R'(s',t'),\ \gamma(s,t) = \alpha^h\gamma'(s',t'), \tag{8a}$$

$$\eta(s,t) = \alpha^{h-2}\eta'(s',t'),\ \zeta(s,t) = \alpha^{h-2}\zeta'(s',t'), \tag{8b}$$

$$\theta(s,t) = \alpha^{h-1}\theta'(s',t). \tag{8c}$$

The desired evolution equation (5a) is also scaling invariant, since it follows from scaling-invariant equations (1) and (4).

At this point Eq. (5a) is an evolution equation for ψ with unknown functions $R(s,t)$ and $\gamma(s,t)$. Through Eq. (6) these functions are functionals of ψ. Equation (6) contains only multiplications of the functions R, γ, η, ζ, and θ with each other and with ψ and its s derivatives. This suggests that $R, \ldots, \theta$ can be chosen as polynomials[7] of ψ and its s derivatives, with τ_0 appearing as a parameter, and the coefficients independent of s or t. Consequently, the evolution equation is also a polynomial in these variables.

The individual terms in a scaling invariant polynomial must each scale in the same way as that polynomial. Each polynomial, R, γ, etc., occurring in Eqs. (5) and (6) can thus be written as a sum of specific terms, with coefficients that follow from (5) and (6). Below we give an example of this procedure for $h = 4$.

The choice $h = 2$ and $h = 3$ respectively yields the nonlinear Schrödinger equation $i\psi_t + 2\psi_{ss} + |\psi|^2\psi = 0$, and the modified Korteweg–de Vries equation $\psi_t + \frac{3}{2}\psi^2\psi + \psi_{sss} = 0$. It is clear by inspection that the scaling (7) leaves these equations invariant. Furthermore, the functions R and γ are invariant: for example, when $h = 2$ we have[2]

$$R = |\psi|^2 - 2\tau_0^2 \text{ and } \gamma = 2i\psi_s - 2\tau_0\psi.$$

Restricting the functional dependence of $G(\alpha)$ to powers of the scaling factor α can only yield evolution equations that are scaling invariant. A more general functional dependence for $G(\alpha)$ allows the Hirota equation, which is not scaling invariant.

Instead of considering evolution equations with dependent variable ψ one can look for evolution equations with dependent variable the phase σ, given in Eq. (7d). This quantity is scaling invariant, just as any functional of σ. Therefore, evolution equations for σ are not restricted to polynomials, unlike evolution equations for ψ. For σ one finds the sine–Gordon equation $\sigma_{st} = \sin\sigma$. This equation also follows when one considers

$\sigma = \int_{-\infty}^{s} \psi(s,t)ds$. However, the scaling exponent h becomes $h = -1$, which is not physically realizable.

We now attempt to find an evolution equation for ψ with scaling exponent $h=4$. Consider the polynomial ζ with scaling power $h-2=2$. Its most general form is a sum of all possible real terms, each with scaling factor α^2:

$$\zeta = c_1 |\psi|^2 + c_2\tau_0^2 + c_3\tau_0(\psi+\psi^*) + ic_4\tau_0(\psi - \psi^*) + c_5(\psi_s + \psi_s^*) + ic_6(\psi_s - \psi_s^*), \quad (9)$$

where the coefficients $c_1 - c_6$ are arbitrary real numbers, to be determined by Eqs. (6b), (6a), and (5a). (A term such as $\tau_0^{-1}\psi_{ss}$ is excluded because τ_0 must be allowed to take any real value, including zero.) Equation (6b) implies

$$\int_{-\infty}^{\infty} \zeta |\psi|^2 ds = 0, \quad (10)$$

for arbitrary ψ. Thus the coefficients $c_1 - c_6$ all vanish, as none of the terms cancel, or can be integrated to zero. Consequently, $\zeta = 0 = \theta$. Since $\theta_s = 0$ and the s dependence in θ enters only through $\psi(s,t)$, θ is independent of ψ, and can only be a function of the constant parameter τ_0. Because θ scales with exponent $h-1=3$ [Eq. (8c)], the most general form of θ is $\theta = a\tau_0^3$, where a is an arbitrary real constant. The functions η and R, with scaling exponents $h-2=2$ and $h=4$ respectively, follow in a similar way from Eq. (6c) as

$$\eta = b|\psi|^2 + c\tau_0^2, \quad (11a)$$

$$R = \tfrac{3}{4}b|\psi|^4 + \tfrac{1}{2}c\tau_0^2|\psi|^2. \quad (11b)$$

Equation (6a) now yields for γ:

$$\gamma = -ib(|\psi|^2\psi)_s - ic\tau_0^2\psi_s + \tau_0 b|\psi|^2\psi + (c-a)\tau_0^3\psi. \quad (11c)$$

Substituting R and γ in Eq. (5a) leads to

$$\psi_t - ib(|\psi|^2\psi)_{ss} - \tfrac{3}{4}b|\psi|^4\psi + 2b\tau_0(|\psi|^2\psi)_s + \tau_0^2[-ic\psi_{ss} + (ib - \tfrac{1}{2}c)|\psi|^2\psi] + (2c-a)\tau_0^3\psi_s + i(c-a)\tau_0^4\psi = 0. \quad (12)$$

The parameter τ_0 is the eigenvalue of the linear scattering equations,[2] and has to be determined by them. Consequently, we must choose the constants a, b, and c such that τ_0 disappears from Eq. (12); hence the trivial result $a=b=c=0$, $\psi_t = 0$.

Proceeding in a similar way the case $h=5$ again yields the trivial equation $\psi_t = 0$. Going beyond $h=5$ is increasingly tedious as the number of terms in equations like Eq. (9) increases rapidly. The number of equations to be satisfied by the coefficients of the various polynomials, however, increases even faster. The likelihood of finding these relatively few coefficients when many more equations than variables must be satisfied seems remote, but cannot rigorously be disproven. The discussion suggests, however, that the connection between helical curve motion and soliton equations found by Lamb[2] is accidental, and cannot be extended to higher order than $h=2$ for the nonlinear Schrödinger and $h=3$ for the modified Korteweg-de-Vries equations.

In conclusion, scaling invariance of vortex motion equations only allows the scaling of Eq. (7c) for the dependent variable in the associated nonlinear evolution equations. Scaling invariance consistent with physically acceptable vortex motion allows the nonlinear Schrödinger equation only. It is furthermore suggested that helical motion can be connected only to those nonlinear evolution equations already found by Lamb.[2]

ACKNOWLEDGMENT

The author acknowledges valuable comments from A. N. Kaufman.

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