Algebraic internal wave solitons and the integrable Calogero–Moser–Sutherland N-body problem

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The Benjamin–Ono equation that describes nonlinear internal waves in a stratified fluid is solved by a pole expansion method. The dynamics of poles which characterize solitons is shown to be identical to the well-known integrable N-body problem of Calogero, Moser, and Sutherland.

Recently, great progress has been made toward solutions of a class of nonlinear evolution equations that exhibit solitons.[1] The remarkable behavior of stability and nonergodicity of these integrable dynamic systems has attracted attention from both physicists and mathematicians. Especially interesting are the wide applications solitons enjoy in various branches of physical theories.[2]

In this Note, we report yet another example of a physical application of this remarkable phenomena. Multisoliton solutions are found explicitly for nonlinear internal waves propagating in a deep stratified fluid. In this case, instead of the well-known Korteweg–deVries equation that describes shallow water waves,[1] Benjamin proposed the following nonlinear integrodifferential equation,[3]

$$2q_t + 2qq_x + Hq_{xx} = 0, \tag{1}$$

where H is the Hilbert transform operator defined by

$$Hq(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{q(z)}{z-x}\, dz. \tag{2}$$

The linear dispersion relation for Eq. (1) is $\omega(k) = \frac{1}{2}k|k|$. This equation was later rederived in a more rigorous way by Ono[4] and is therefore named the Benjamin–Ono equation. Benjamin[3] found a single solitary wave solution which has a Lorentzian profile,

$$\begin{aligned} q^{(1)}(x,t) &= \frac{i}{(x - vt - x_0) + i/2v} + \text{c.c.} \\ &= \frac{1/v}{(x-vt-x_0)^2 + (1/2v)^2} \quad (v>0), \end{aligned} \tag{3}$$

with constant velocity v, height $4v$, and width $1/v$. Benjamin[3] and also Davis and Acrivos,[5] who performed experiments in water tanks had observed these solitary waves. They also noted that two such solitary waves come out of collisions from each other unscathed like solitons in the Korteweg–deVries equation. Recently, Case and Rosenbluth[6] have found that the single solitary wave solution (3) is stable against linear perturbations. Meiss[7] and Pereira integrated (1) numerically and confirmed the nonlinear stability of the solitary wave solutions (3) by following collisions of these solitary waves. They also observed that a general initial wave profile almost always breaks up into these solitary waves. Joseph[8] applied Hirota's method showing the existence of at least a two-solitary wave solution. All these facts strongly suggest that these solitary waves should be solitons. We will show in the following that this is indeed true. Equation (1) is integrable analytically with multi-soliton solutions.

We apply the pole expansion method of Airault *et al.*[9] and also the Choodnovsky brothers[10] to Eq. (1). From Eq. (3) we note that the single soliton solution has a pair of poles in the complex plane. They are complex conjugate symmetric with respect to the real axis. This suggests that we express the general N-soliton solutions as superpositions of N pairs of poles a_j and a_j^* $(j=1\ldots N)$

$$q^{(N)}(x,t) = \sum_k \frac{i}{x-a_j} + \text{c.c.} \tag{4}$$

Note that a_j is chosen to lie below the real axis, and also $a_j \neq a_k$ for $j \neq k$. It is then easy to see that

$$H\left(\frac{1}{x-a_j}\right) = \frac{i}{x-a_j} \quad . \tag{5}$$

Substituting Eqs. (4) and (5) into (1) and noting that

$$\sum_{k\neq j}\sum_{j=1}^{N} \frac{1}{(x-a_j)(x-a_k)^2} = \sum_{k\neq j}\sum_{j=1}^{N} \frac{1}{(a_j-a_k)}\frac{1}{(x-a_k)^2} \tag{6}$$

we obtain $2N$-coupled ordinary differential equation of motions for these N pairs of poles

$$i\dot{a}_j = \sum_{k\neq j} \frac{1}{a_k - a_j} + \sum_k \frac{1}{a_j - a_k^*} \quad . \tag{7}$$

This is remarkable since the number of unknowns a_j and a_j^* equals the number of equations. Equation (7) is self-consistent, therefore, it implies the indestructibility of these solitons. By following motions of these poles, we are actually following the motion of solitons. We can neither create nor destroy a pole and therefore, the identities of these solitons are preserved.

It is even more remarkable that Eq. (7) can be embedded into an integrable Hamiltonian system, the well-known Calogero–Moser N-body problem with pairwise inverse square potential.[11,12] By taking the time derivative of Eq. (7), and some straightforward algebra, we can reduce Eq. (7) to the form

$$\ddot{a}_j = 2\sum_{k\neq j} \frac{1}{(a_j-a_k)^3} \quad . \tag{8}$$

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This is the Calogero–Moser N-body system, known to be integrable by satisfying Lax's integrability condition.[11] Equation (8) is equivalent to the operator equation

$$L_t = [A, L]$$

with

$$L_{ij} = \delta_{ij}\dot{a}_j + i(1-\delta_{ij})/(a_i - a_j), \tag{9}$$

$$A_{ij} = -i\delta_{ij}\sum_{k\neq i}(a_i - a_k)^{-2} + i(1-\delta_{ij})(a_i - a_j)^{-2}. \tag{10}$$

Substituting Eq. (7) into Eq. (9), we immediately obtain the Lax representation for Eq. (7). Therefore, in addition, to knowing that solitons are indestructible, we have also established that their motions are non-ergodic. They are determined smoothly from the initial data (in the present case, the initial position of the poles). To obtain explicit solutions of Eq. (7), we derive two sets of quantities from the Lax operator L, namely, the set of N motion invariants (action variables),

$$I_n \equiv \mathrm{Tr}(L^n), \tag{11}$$

such that $dI_n/dt = 0$, and another set of N quantities (angle variables) obtained by Olshanetsky and Perelomov[13]

$$J_n \equiv \mathrm{Tr}(BL^{n-1}), \tag{12}$$

with $B_{ij} \equiv a_i\delta_{ij}$. It is easily shown that

$$B_t = [A, B] + L$$

and therefore

$$dJ_n/dt = I_n \text{ or } J_n(t) = I_n(0)\,t + J_n(0). \tag{13}$$

Equations (11) and (13) then explicitly yield solutions of $a_j(t)$. For example, in the case $N=2$, the two invariants are

$$I_1 = \dot{a}_1 + \dot{a}_2 \quad ,$$
$$I_2 = \dot{a}_1^2 + \dot{a}_2^2 + [2/(a_1 - a_2)^2]. \tag{14}$$

From Eq. (13), we also have

$$J_1 = a_1 + a_2 = I_1 t + J_1(0) \quad ,$$
$$J_2 = a_1\dot{a}_1 + a_2\dot{a}_2 = I_2 t + J_2(0), \tag{15}$$

or

$$a_1^2 + a_2^2 = I_2 t^2 + 2J_2(0)t + a_1^2(0) + a_2^2(0).$$

Therefore, a_1 and a_2 are solutions of the algebraic equation

$$X^2 - J_1 X + \tfrac{1}{2}[J_1^2 - I_2 t^2 - 2J_2(0)t - a_1^2(0) - a_2^2(0)] = 0. \tag{16}$$

Detailed analyses of this solutions have been studied by Calogero[12] and we shall not duplicate them here. Instead, we give an explicit N-pole solution by Eq. (8) first obtained by Olshanetsky and Perelomov.[13] They showed that eigenvalues of the operator

$$M(t, t_0) \equiv B(t_0) + (t - t_0)\,L(t_0), \tag{17}$$

coincide with the poles $a_l(t)$ for $l = 1, 2 \ldots N$. Since $B(t_0)$ and $L(t_0)$ are explicit functions of t_0, the initial time, we thus have the eigenvalues of M as the explicit solutions of the initial value problem for Eq. (8).

We now turn to solutions of Eq. (8) in the case of periodic boundary condition. The period is chosen to be $L \equiv 2\pi/\alpha$.

The corresponding pole expansion for periodic solutions is now given by

$$q(x,t) = \sum_k i\alpha\cot[\alpha(x - a_k)] + \mathrm{c.c.} \tag{18}$$

and the pole dynamics are governed by

$$i\dot{a}_j = \alpha\sum_{k\neq j}\cot[\alpha(a_k - a_j)] + \alpha\sum_k \cot[\alpha(a_j - a_k^*)]. \tag{19}$$

It can be embedded into the integrable Moser–Sutherland[14] many-body problem,

$$\ddot{a}_j = 2\alpha^3\sum_{k\neq j}\csc[\alpha(a_j - a_k)]\cot^2[\alpha(a_j - a_k)]. \tag{20}$$

The corresponding Lax operators are given by Moser[11]

$$L_{ij} = \delta_{ij}\dot{a}_j + i(1-\delta_{ij})\,\alpha\cot[\alpha(a_i - a_j)], \tag{21}$$

$$A_{ij} = -i\delta_{ij}\alpha^2\sum_{k\neq i}\csc^2[\alpha(a_i - a_k)] + i(1-\delta_{ij})\csc^2[\alpha(a_i - a_j)]. \tag{22}$$

We do not know the corresponding B and M operators in the periodic case. However, we can still obtain the N-motion invariants from Eq. (11). Explicit solutions can thus be obtained algebraically.

In conclusion, we have solved Eq. (1) with both the infinite and periodic boundary conditions by a pole expansion method. The N-soliton solutions thus obtained is represented by N pair of poles in the complex plane. Their motions are nonergodic and are governed by the well-known integrable Calogero–Moser–Sutherland N-body problem.

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