Damped double solitons in the nonlinear Schrödinger equation

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The nonlinear Schrödinger equation modified by a damping term is investigated numerically for initial conditions other than single solitons. With damping, colliding solitons still pass through each other, but the breather can change qualitatively into two continuously interacting but separated solitons. These results are consistent with a slow change in the inverse scattering eigenvalues due to the damping.

I. INTRODUCTION

The nonlinear Schrödinger equation, Eq. (1), arises as the envelope equation of a dispersive wave system which is almost monochromatic and weakly nonlinear.[1] For example, two plasma heating problems of current interest are approximated by this equation, in their nonlinear stage, viz., (i) Langmuir turbulence when the background plasma is assumed in equilibrium with the ponderomotive pressure from the high-frequency fields,[2,3] and (ii) a nonlinear stage of the mode-converted wave in lower hybrid heating of large tokamaks.[4]

When such a wave heats (transfers energy to) the particles of the plasma, a dissipation term appears in the nonlinear Schrödinger equation. Since the heating is slow, the dissipation term is small and can be considered as a perturbation that, hopefully, leaves some qualitative properties of the solution unchanged. In Langmuir turbulence, for instance, the dissipation is wavenumber-dependent Landau damping,[2,3] while for the lower hybrid wave the damping is more difficult to obtain.[5]

The nonlinear Schrödinger equation is one of a class of exactly solvable evolution equations. These equations have various properties in common, notably stable nonlinear wave solutions called solitons, and an infinite set of conservation laws.[6-8] It is well known[8] that a large enough initial condition in such an equation typically evolves into solitons. Thus, it is necessary to study the effect of damping on single solitons, but this is not sufficient; for a more complete understanding one must find out how more general initial conditions behave[9] under damping.

In a previous paper[10] we treated single solitons with damping as perturbation, and established that single solitons damp in substantial agreement with a simple treatment based on their invariant shape and the first conservation law.[11-13] We discussed, for example, the influence on the damping rate of the exponent b in the damping law $\gamma_k = |k|^b$, and showed that the damping rate is a constant only for $b=0$ and $b=2$. Such a comparison between numerical computations and analytical considerations provides one example of construction and verification of possible soliton perturbation theories; after all, damping is just one particular perturbation.

A complete perturbation theory for soliton equations should not only predict the evolution of single solitons, but ideally should be able to treat arbitrary initial conditions. In an unperturbed soliton equation, every initial condition develops into a background (radiation), which is supposed to disperse away and become unimportant over time, and into solitons, which stay around permanently (but even this unperturbed solution can usually not be calculated analytically).

The final solitons may have unequal velocities, in which case they exhibit pairwise collisions, or they may have equal velocities (in some equations such as the nonlinear Schrödinger equation), in which case they form a nonlinear superposition called a breather.

In the last few years various soliton perturbation theories[14-16] have been developed. These theories assume that single solitons keep their shape, but adiabatically change their parameters (amplitude and velocity). For more than one soliton, they yield nonlinear relations between all parameters of the constitutent solitons, including the intersoliton distance. These relations contain coefficients, spatial integrations over the soliton shape multiplied by the perturbation, that are almost intractable for other than single solitons. Therefore, we have not been able to extend our detailed analytical checkup on perturbed single solitons in Ref. 10 to double solitons. Instead, we attempt to numerically confirm the validity of one particular soliton perturbation theory based on the conservation laws. This approach is especially convenient for damping, and gave good results with relatively little effort for single solitons.

For the breather, a superposition of two solitons, we need two parameters: hence, besides the first we must use the third conservation law, in which the soliton parameters enter nonlinearly. For our purpose, the numerical verification of the two-time-scale assumption common to all soliton perturbation theories, this nonlinearity and the analytically prohibitive space integrations over soliton shape and perturbation present no special difficulty.

This paper, then, extends our previous work[10] on

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single solitons to the simplest two-soliton cases, namely, to collisions of two equal solitons with opposite velocities, and to the simplest breather. The perturbation is again a simple damping of each Fourier mode with its own damping rate $\gamma_k = \epsilon|k|^b$. We concentrate on the two simplest dampings, namely, collisional damping, $b=0$, and the damping $b=2$, which introduces a small imaginary part in the coefficient of the dispersive term, but, in contrast with the comparison with an analytical prediction, here we compare to another numerical computation that uses the two-time-scale assumption and the conservation laws.

In Sec. II we briefly discuss the inverse scattering transform and its eigenvalues, and give the relevant data on damping of single solitons. In Sec. III we treat colliding solitons. In Sec. IV we numerically study the damped breather in some detail, and show that its evolution is consistent with a two-time-scale assumption. In Sec. V we present our conclusions, including the generalization of these results to soliton perturbation theories.

II. BASICS

Our nonlinear Schrödinger equation has the form

$$iq_t + q_{xx} + 2|q|^2 q = 0\,, \qquad (1)$$

where $q(x, t)$ is a complex function of the real variables t (time) and x (space). A single soliton has the form

$$q_s(x, t) = 2\eta\,\mathrm{sech}[2\eta(x - 4\xi t)]\exp(i\theta)\,, \qquad (2)$$

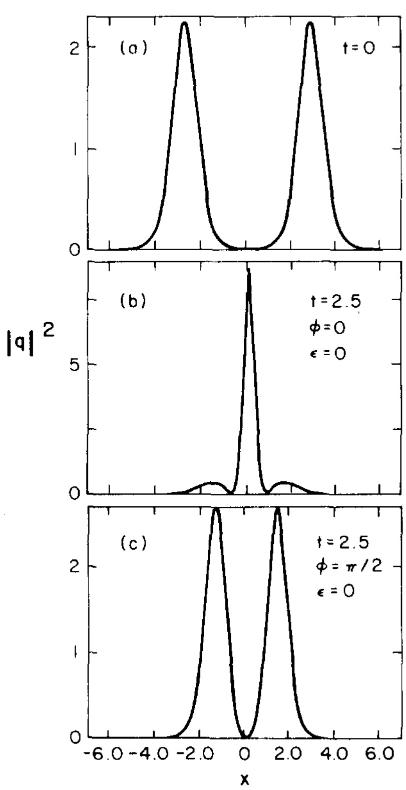

FIG. 1. Undamped solition collision. Parameters are $2\eta = 1.5$ and $V = \pm 0.75$. (a) Initial condition, (b) collision stage at $t = 2.5$ with no initial phase difference $\phi = 0$, and (c) collision stage at $t = 2.5$ with initial phase difference $\phi = \frac{1}{2}\pi$.

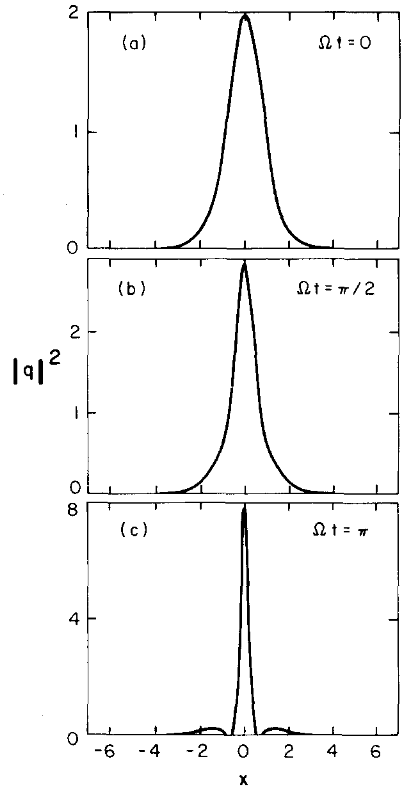

FIG. 2. The undamped breather at three stages of its periodic development: (a) $t=0$, (b) $\Omega t = \frac{1}{2}\pi$, and (c) $\Omega t = \pi$.

with phase $\theta = 2\xi x - 4(\xi^2 - \eta^2)t$. The parameter η determines the amplitude and inverse width of the soliton, and 4ξ is the velocity. This notation is not the simplest and deviates from previous use, but is appropriate for the inverse scattering transform whose notation we will employ.

Equation (1) has an infinite set of conservation laws. The first few are reminiscent of a particle mass in quantum mechanics,

$$I_1 = \int |q|^2\, dx\,, \qquad (3)$$

the momentum, $I_2 = \int (q^* q_x - q q^*)\, dx$, and the energy,

$$I_3 = \int \; |q_x|^2 - |q|^4\, dx\,. \qquad (4)$$

(all integrations are over the whole real axis $-\infty < x < \infty$). The higher conservation laws have no direct physical meaning, and are more complicated.

The inverse scattering transform shows that the complete nonlinear evolution of arbitrary initial conditions can be understood in terms of solitons, and a nonsoliton part called radiation. The radiation part is complicated, and we choose not to treat it here.[16]

The solitons each correspond to two parameters, the real part ξ and the imaginary part η, of the eigenvalue (ζ) in some linear scattering problem. In general, it is difficult to find the eigenvalue for a given initial condition, but one can write down, explicitly, a full solution that corresponds to given eigenvalues, usually a complicated combination of exponentials which depend on the eigenvalues and on additional parameters that correspond to initial intersoliton distances and phases. The solutions are plotted in Figs. 1 and 2 for the two special cases we consider. Figure 1 shows $|q|^2$ for colliding solitons with initial condition

$$q(x, t=0) = q_s(x - x_0, t=0) + q_s(x + x_0, t=0)\exp(i\phi) \tag{5}$$

and q_s from Eq. (2). The parameters are $x_0 = 3$, or an intersoliton distance 6, amplitude $2\eta = 1.5$, and velocity $4\xi = \pm 0.75$. The initial phase difference ϕ is not visible in Fig. 1(a) for the initial condition, but the collision stage is very different for the two cases: for $\phi = 0$, in Fig. 1(b), there is a large peak due to soliton overlap while for $\phi = \frac{1}{2}\pi$, in Fig. 1(c), the solitons are bouncing off of each other. The final state is similar to Fig. 1(a). The full solution for the breather is

$$q(x,t) = 4i\,|\eta_1^2 - \eta_2^2|\,\frac{\eta_1 \cosh 2\eta_2 x \exp(i\Omega_1 t) + \eta_2 \cosh 2\eta_1 x \exp(i\Omega_2 t)}{2(\eta_1 - \eta_2)^2 \cosh 2\eta_2 x \cosh 2\eta_1 x + 4\eta_1\eta_2\{\cosh[2x(\eta_1 - \eta_2)] + \cos\Omega t\}}\ , \tag{6}$$

where $\Omega_1 = 4\eta_1^2$, $\Omega_2 = 4\eta_2^2$, and $\Omega = \Omega_2 - \Omega_1$. Notice that the breather amplitude $|q|^2$ is purely periodic, with the only time dependence entering through two occurrences of $\cos(\Omega t)$. Various stages for the breather evolution are given in Fig. 2, for the particular choices of eigenvalues $\eta_2 = \frac{3}{2}$ and $\eta_1 = \frac{1}{2}$. At $t = 0$ in Fig. 2(a), $q(x, t=0) = 2\,\mathrm{sech}(x)$, a single soliton whose amplitude is multiplied by two. This narrows slowly to the form plotted in Fig. 2(b) at $\Omega t = \frac{1}{2}\pi$. Then, the narrowing accelerates to the contracted breather stage given in Fig. 2(c) at $\Omega t = \pi$. Notice the amplitude of the peak, and the large values of the derivatives q_x. Breathers with eigenvalues other than $\frac{3}{2}$ and $\frac{1}{2}$ are qualitatively similar at $\Omega t = \pi$, but at $t = 0$ these other breathers can show a double-humped shape.

For the two cases that we consider with only two eigenvalues, the values of the conserved quantities are directly related to the eigenvalues:

$$I_1 = 4(\eta_1 + \eta_2)\ , \tag{7a}$$

$$I_2 = 16(\eta_1\xi_1 + \eta_2\xi_2)\ , \tag{7b}$$

$$I_3 = 16(\eta_1\xi_1^2 - \tfrac{1}{3}\eta_1^3 + \eta_1\xi_2^2 - \tfrac{1}{3}\eta_2^2)\ . \tag{7c}$$

For colliding solitons $\eta_1 = \eta_2 = \eta$ and $\xi_1 = -\xi_2 = \xi$: hence, $I_2 = 0$, and $I_3 = 32(\eta\xi^2 - \frac{1}{3}\eta^3)$. The breather has eigenvalues with real parts equal to zero, but unequal imaginary parts: again, $I_2 = 0$ and $I_3 = -\frac{16}{3}(\eta_1^3 + \eta_2^3)$. Thus, the eigenvalues can be found directly from the values of the conserved quantities, by a simultaneous solution of a first order and third order polynomial equation.

Now, we introduce damping by adding an extra term to Eq. (1), $i\,\mathrm{FT}^{-1}(\gamma_k q_k)$, where FT^{-1} denotes the inverse Fourier transform and k is the wavenumber. As discussed in Ref. 10, in the absence of nonlinearity this term would damp each Fourier mode, $q_k = 1/2\pi \int q(x) \times\exp(-ikx)dx$, with its own damping decrement λ_k.

We consider the two simplest cases of the model damping $\gamma_k = \epsilon|k|^b$, namely $b = 0$ (collisional damping) and $b = 2$, as a rough but convenient approximation to Landau damping. For $b = 0$, Eq. (1) acquires an extra term and becomes

$$iq_t + i\epsilon q + q_{xx} + 2|q|^2 q = 0\ , \tag{8}$$

while the case $b = 2$ just changes the coefficient of the dispersive term to a complex number,

$$iq_t + (1 - i\epsilon)q_{xx} + 2|q|^2 q = 0\ . \tag{9}$$

These seemingly innocuous changes in the equation have various and sometimes dramatic effects. Firstly, with the extra terms the inverse scattering transform does not apply, which is why we must use perturbation theory. Secondly, the quantities that are conserved under Eq. (1) are no longer conserved. The equations for these changes[10] become, for the case $b = 0$:

$$dI_1/dt = -2\epsilon I_1\ , \tag{10a}$$

and

$$dI_3/dt = -2\epsilon \int |q_x|^2 - 2|q|^4\,dx\ , \tag{10b}$$

$$dI_3/dt = -2\epsilon\left[3I_3 - \int 2|q_x|^2 - |q|^4\,dx\right]. \tag{10c}$$

For $b = 2$ we obtain

$$dI_1/dt = -2\epsilon \int |q_x|^2\,dx\ , \tag{11a}$$

and

$$dI_3/dt = -2\epsilon \int |q_{xx}|^2 - 2[(|q|^2_x)^2 + 4|q|^2|q_x|^2]\,dx\ , \tag{11b}$$

$$dI_3/dt = -2\epsilon \int |q_{xx}|^2 + |q|^2(q^* q_{xx} + q q^*_{xx})\,dx\ . \tag{11c}$$

Even though I_1 and I_3 change in time proportional to ϵ we will still refer to them as conserved quantities. The invariant I_2 always remains zero for symmetric initial conditions.

We notice that Eq. (10a) describes an exponential decay for I_1 irrespective of the solution $q(x, t)$, but that the others do depend on the solution in some complicated way: The right-hand sides of Eqs. (10b)–(11b) do not reduce to combinations of conserved quantities. Compare, for example, Eq. (10b) or (11a) with Eq. (3b), or Eq. (11b) with the next-higher conservation law[6]

$$I_5 = \int dx[\,|q_{xx}|^2 - (|q|^2_x)^2 - 6|q|^2|q_x|^2 + 2|q|^6]\,.$$

However, the right-hand sides are constant when the solution keeps a constant shape, that is, for a single soliton. For double solitons, there is generally a strong dependence of the solution on time, and hence the expressions in Eqs. (10) and (11) are also time dependent. In this connection we recall from Ref. 10 that the decrease of I_1 and I_3 is consistent with the assumption of a stationary sech-shaped soliton with decreasing amplitude parameter. For $b=0$, this is easy to see in Eq. (10c): The integral is zero, and I_3 is proportional to the third power of I_1. For double solitons this conclusion is no longer true, because its proof hinges on the explicit functional form, sech(x), of the single soliton.

Although it may not be apparent from Eqs. (11), we believe that the case $b=2$ is particularly simple, partly on the basis of Eq. (9) but mostly because in this case an exact stationary soliton shape can be found.[10] For other values of the damping exponent b, however, including $b=0$, there are shape changes of the soliton to second order in ϵ: these have a time-dependent effect on the damping rates that is readily noticeable.

For our purpose, it is convenient to construct a simple perturbation theory on the basis of the conservation laws. We adopt a conventional two-time-scale assumption: the constant parameters of the unperturbed problem, the inverse scattering eigenvalues in this case, change slowly in time when the perturbation, i.e., damping, is introduced. This assumption is very successful for single solitons, or single eigenvalues, as shown in Ref. 10. With damping, single solitons approximately keep their shape, which reflects the continuing balance between nonlinearity and dispersion, but the solitons adapt their amplitude and width to agree with the change in the first conserved quantity.

Now, we generalize to more eigenvalues, in which case the solutions, the breathers, are not stationary; but, there is still a balance between nonlinearity, dispersion, and now also the time derivative (the time derivative does enter the single soliton balance but in a trivial way). This balance is only slightly affected by the damping which, however, causes the conserved quantities to change according to the exact equations (10) and (11). In some complicated nonlinear way the conserved quantities then determine the eigenvalues through Eq. (7), at least when the number of variables equals the number of equations. In principle, there is an infinite number of equations such as Eq. (7), and we are faced with an overdetermined system. For a single soliton, we know that the change of a single variable is consistent with at least two of these equations.[10] In view of the exact solution for damping exponent $b=2$, the damping single soliton is even consistent with an infinity of equations (7).

Generalizing to double solitons we determine the eigenvalues by the minimum number of conservation laws, and ignore the higher ones.

The two time scales in our approach are then: (i) the slow time scale of order ϵ due to the damping, and (ii) the natural time scale determined by the nonlinear Schrödinger equation (1) and the initial condition. For breathers, this time scale is $\pi/[4(\eta_2^2-\eta_1^2)]$, the period Ω^{-1} in Eq. (6). For stationary single solitons the time scale degenerates to infinity.

Just as other soliton perturbation theories, the present approach is analytically prohibitively complicated for breathers, especially for the damping exponent $b=2$ when Eq. (11) applies. Since the right-hand sides are not reducible to conserved quantities, they must be evaluated with the explicit functional form, Eq. (6). Numerically, however, the integrations are straightforward.

Our numerical procedure is then as follows. We numerically compute the integrals in Eqs. (10) or (11) at a particular time for given eigenvalues η_1 and η_2 using Eq. (6). Then, we change the conserved quantities over a small timestep Δt according to Eqs. (10) and (11), and recompute the eigenvalues at the next time $t+\Delta t$ from Eq. (7). Note that in Eq. (6) we should replace the time dependence Ωt by $\int^t \Omega(t')\,dt'$ in the spirit of the two-time-scale assumption.

Iteration of this procedure then gives a time evolution to be compared in Sec. IV with the damped breather as computed from the nonlinear Schrödinger equation with damping included, Eq. (8) or (9).

It is obvious how to extend this procedure, in principle, to cases with more than two eigenvalues. It is only moderately clear how to include intersoliton distances and phases: the velocities $V(t)=4\xi(t)$ can be determined from the second conservation law I_2, whose time dependence we have not written down. An additional temporal integration then determines the each soliton position. This procedure works well for single solitons.[10]

However, it is not clear how to include radiation.[16] We do not consider this an important drawback, because things are sufficiently complicated already with only eigenvalues, the double solitons treated in this paper.

III. COLLIDING SOLITONS

A computation to determine whether two solitons survive their nonlinear collision is a numerically convenient first step to possible exact solvability of the underlying equation. Therefore, in this section we show how damping affects a soliton collision. The initial condition (5), shown in Fig. 1(a), consists of two spatially separated equal solitons with opposite velocities, and a phase difference ϕ. With no damping the solitons would collide, have a complicated nonlinear interaction shown in Fig. 1, and would emerge unscathed.

We find no qualitative change in this behavior when damping is included. Figures 3(a) and 3(b) show the collision stage with damping strength $\epsilon=0.1$ and $b=2$ for $\phi=0$ and $\phi=\pi$, for comparison with Fig. 1(b) and 1(c). Single solitons with $b=2$ decrease their velocity on damping, and consequently the solitons reach each other later than with no damping. For $\phi=0$, their collision still generates a peak, shown in Fig. 3(a) at $t=3.0$, with amplitude reduced by damping, while for $\phi=\pi$ the

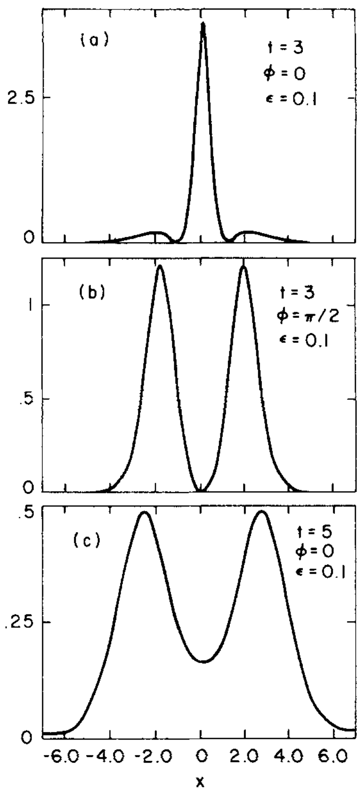

FIG. 3. Damped soliton collision with damping parameters $b=2$, $\epsilon=0.1$ at $t=3$ and $t=5$. (a) Collision stage at $t=3.0$ with initial phase difference $\phi=0$. (b) Collision stage at $t=3$ with initial phase difference $\phi=\pi/2$. (c) Postcollision stage at $t=5$ for $\phi=0$.

solitons still bounce off each other and roughly remain single solitons with changing velocities. Figure 4 shows $I_1=\int|q|^2\,dx$, which changes according to Eq. (10a) and therefore reflects the shape changes during collision. At first the solid curve, I_1 for $\phi=0$, flattens as the solitons widen in their pull toward each other, and the subsequent steep decline around $t=2.75$ corresponds to the peak at $x=0$ of Fig. 3(a). In contrast, the decrease in I_1 for $\phi=\pi$, the dashed curve, is very regular because there are no appreciable shape changes in the collision. After the collision the solitons emerge smaller and slower but otherwise unaltered; but because the solitons are wider and the velocity has decreased, they are still overlapping at the end of the run, as seen in Fig. 3(c) for $\phi=0$.

The case of collisional damping, $b=0$, is unexciting

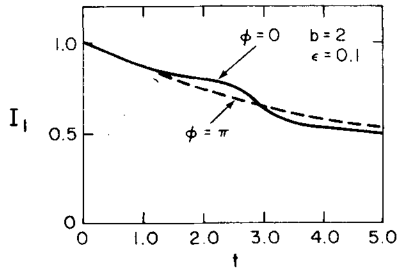

FIG. 4. $I_1=\int|q|^2\,dx$ versus time for the cases of Fig. 1.

because now I_1 does not depend on soliton shape and just shows exponential decay, while only I_3 changes slightly in time similar to I_1 for the case $b=2$. The solution in x space is a widening and diminishing version of the undamped case, with no change in the velocity.

These computations suggest that it is reasonable to make a two-time-scale expansion of soliton dynamics. Unfortunately, the intersoliton distance, which codetermines the soliton shape, does not appear in the conservation laws, but must be approximated by a temporal integration of the velocity. We avoid this complication in our study of the breather, where this intersoliton distance appears to remain zero.

IV. THE DAMPED BREATHER

What is the influence of damping on the characteristics of the breather? We recall that the breather in the absence of damping, shown in Fig. 2, is a purely periodic solution with period $T=2\pi/(4\eta_2^2-4\eta_1^2)$: with the imaginary parts of the inverse scattering eigenvalues $\eta_2=\frac{3}{2}$ and $\eta_1=\frac{1}{2}$, $T=\frac{1}{4}\pi$. At time $t=0$ (modulo T) the breather reduces to a soliton multiplied by two, $q(x,t=0)=2\,\mathrm{sech}(x)$. At half-periods, $t=\frac{1}{2}T$, modulo T, the breather contracts to a very high and consequently narrow state with large derivatives.

When collisional damping is introduced and Eq. (8) applies, the invariant I_1 is exponential in time, in agreement with Eq. (10a). Figure 5 shows this evolution, and the behavior of the eigenvalues η_1 and η_2 as computed from the conservation laws. The eigenvalues are symmetric around $I_1=4(\eta_1+\eta_2)$, and have an overall exponential decay with the waviness superimposed. The waviness is due to the enhanced decrease of I_3 when the breather is in the contracted state. The derivatives are then large [see Fig. 2(c)], and therefore, the right-hand side of Eq. (10b) is large.

In x space the breather approximately returns to its original shape, just like an undamped breather, but with decreased amplitude. This explains the increase in the separation between the waviness of Fig. 5.

The numerical results are rather more complicated for the case $b=2$, which for a single soliton was the simplest. In Fig. 6(a) we plot the invariant I_1 (solid line) and the eigenvalues (dashed lines) as functions of

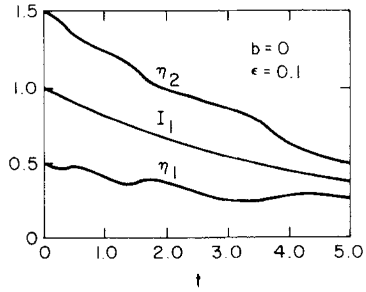

FIG. 5. $I_1=\int|q|^2\,dx$ and the eigenvalues η_1 and η_2 versus time for a damped breather, with damping parameters $b=0$ and $\epsilon=0.1$.

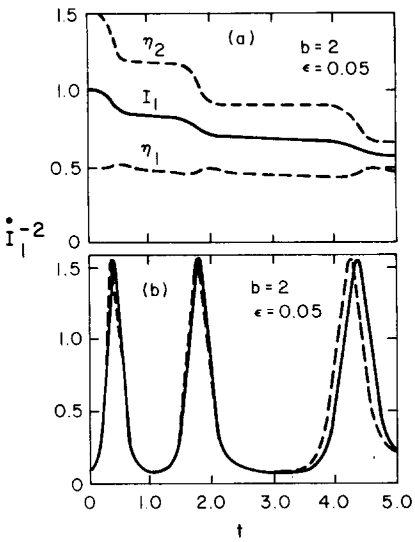

FIG. 6. (a) $I_1 = \int |q|^2\, dx$ and the eigenvalues η_1 and η_2 for a damped breather, with damping parameters $b=2$ and $\epsilon = 0.05$. (b) The soliton shape measure dI_1^{-2}/dt versus time from Eq. (9) (solid line) and from the two-time-scale assumption (broken line).

time. Again, we notice the approximately periodic decrease of I_1, but now there is a strong step-like dependence of I_1 on time. The relatively slow decrease of I_1 between the "steps" corresponds to full periods, when the undamped breather would be sech-shaped, while the "steps" themselves come from the contracted breather state. However, when we measure the elapsed time between two successive steps of I_1 and correct for the increase in time scale due to the decreased amplitude, we find no exact periodicity. A qualitative reason for this apparent lack of periodicity is evident in the space plots of $|q(x)|^2$ shown in Figs. 7(b) and 7(c). With damping the breather does not return to its original shape, but instead it seems to split in two soliton-like shapes that overlap only moderately. Thus, the breather period increases because the mutual attraction between the two constituent solitons in the breather diminishes as their overlap decreases.

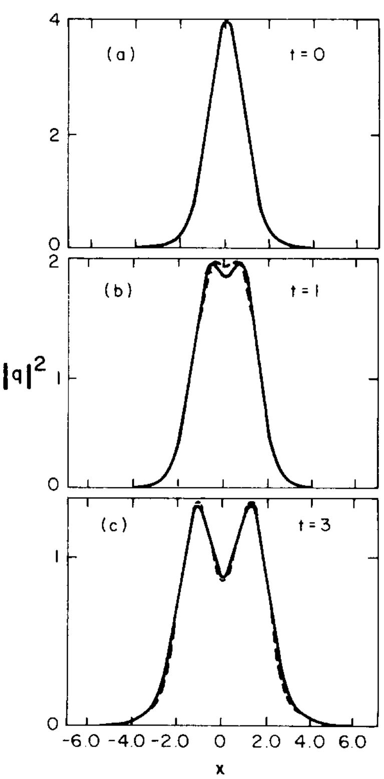

FIG. 7. Shapes of a damped breather, with damping parameters $b=2$ and $\epsilon = 0.05$. (a) Initial condition $q(x) = 2\ \text{sech}\ (x)$, (b) at $t=1$, from Eq. (9) (solid line) and from the two-time-scale assumption (dashed line), (c) at $t=3$.

In Ref. 10 we used the time derivative of I_1^{-2} as a measure of soliton shape. This quantity is plotted in Fig. 6(b). The various humps, which correspond to the steep decline in I_1 but are normalized with I_1^3, have nearly equal maxima and shapes. This seems to indicate that at least the contracted breather is approximately scaling invariant. With increasing time there is, however, a definite increase in amplitude and a widening of these now slightly asymmetric peaks.

The increase in period and the shape changes of the breather can be understood from the eigenvalues η_1 and η_2, given in Fig. 6(a) by the dashed lines. The smaller eigenvalue hovers around the initial value $\frac{1}{2}$, but the larger eigenvalue, initially $\frac{3}{2}$, decreases with similar but larger steps than those of I_1 (I_3, not shown, has an even stronger time dependence). Thus, the difference between the eigenvalues decreases, and hence the period increases, since $T \propto (\eta_2^2 - \eta_1^2)^{-1}$ for an undamped breather. The double-humped shape of the damped breather is less easily understood, because the analytical formula is complicated, but it can easily be shown numerically that such shapes indeed originate from two eigenvalues that are close together. The dashed lines in Figs. 7(b) and 7(c) give a plot of Eq. (8) with approxi-

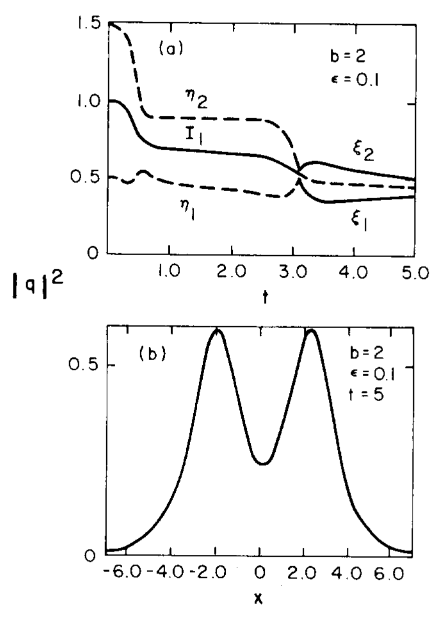

FIG. 8. (a) $I_1 = \int |q|^2\, dx$ and the imaginary part of the eigenvalues, η, and the real parts of the eigenvalues, ξ, for a damped breather with damping parameters $b=2$ and $\pi = 0.1$, (b) the breather shape at $t=5$.

mately the eigenvalues at that particular time. These dashed curves are further discussed later on.

At later times than shown here, or for larger dampings, the two imaginary parts η_1 and η_2 of the eigenvalues ζ coalesce. At this point the eigenvalues acquire real parts ξ, which means that the constituent solitons have obtained a velocity and separate asymptotically.

This is demonstrated by Fig. 8(a) for larger damping strength $\epsilon = 0.1$. Initially, the eigenvalues and I_1 behave qualitatively as in Fig. 4 for $\epsilon = 0.05$, but at $t = 3.1$ they coalesce and develop real parts ξ. The magnitude of ξ is proportional to the velocity; this velocity is indicated by the difference between the broken lines, the sum of the eigenvalues I_1, and the solid line. The solitons at $t = 5$ are given in Fig. 8(b): They are clearly well separated, and could very well separate completely for larger times. Whether they actually separate is of little practical importance, because the soliton amplitude decreases rapidly for this damping strength $\epsilon = 0.1$.

As long as the η's differ, the eigenvalue ζ cannot develop a real part for the following reason. Suppose that with $\eta_1 \neq \eta_2$ there would be a real part ξ to ζ_1 and ζ_2 at some particular time t'. The ζ's must be of opposite sign, because of $I_2 = 16\,(\eta_1 \xi_1 + \eta_2 \xi_2) = 0$. Now remove the damping for times greater than t', so that Eq. (1) is again satisfied and the given initial condition evolves in such a way that the eigenvalues remain constant. Because of their opposite velocities, the solitons that constitute the breather must separate to eventually form two disjoint solitons that are unequal, since the η's differ. Thus, we would have an asymmetric final solution. This cannot happen because the initial condition $q(x, t=0) = 2\,\mathrm{sech}(x)$ and both Eqs. (1) and (9) are invariant under reflection $x \to -x$.

The solid line in Fig. 9 shows the eigenvalues plotted against each other to further clarify the shape changes of the breather. The important parameter here is the ratio of eigenvalues η_2/η_1. The eigenvalue η_1 for $b = 2$, in Fig. 8(b), oscillates in a narrow band around $\frac{1}{2}$ while η_2 decreases until both eigenvalues ζ obtain real parts. For $b = 0$, Fig. 8(a), the two eigenvalues oscillate around the straight line $\eta_2 = 3\,\eta_1$. Thus, in this case the ratio of eigenvalues η_2/η_1 remains at its initial value 3, and corroborates the recurrence of the initially sech-shaped breather. Note that the oscillation amplitude of the waviness remains approximately constant in time for $b = 2$, Fig. 9(b), but increases for $b = 0$, Fig. 7(a). As explained in Ref. 10, this increase is due to the growth of the damping term relative to the other terms for decreasing soliton amplitude. In contrast, when Eq. (9) applies, the case $b = 2$, the damping term is always ϵ times the dispersion, and the oscillation amplitude remains constant

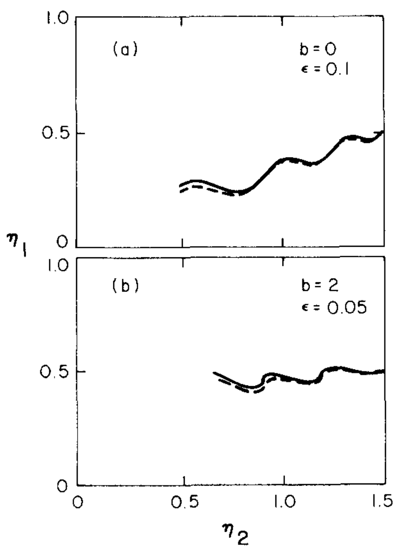

FIG. 9. The breather eigenvalues η_1 and η_2 plotted against each other, from Eq. (9) (solid line) and the two-time-scale assumption (broken line). (a) From Fig. 3, (b) from Fig. 4.

All these results are obtained from a numerical solution of Eqs. (8) and (9), and the eigenvalues are computed from values of the conservation laws which, as we observe, change in time with large steps. Now, we must establish the validity of the two-time-scale assumption that forms the basis of the available perturbation theories, including our own in Sec. II.

Ideally, we should do this by comparison of our results with analytical formulae of the kind written down formally in Refs. 14–16, or in Sec. II. The analytical evaluation of such expressions for the breather is, however, prohibitively complicated and unrevealing. Therefore, instead we compare with an additional numerical computation which assumes that at time t there exists a breather solution of the form given in Eq. (6), with eigenvalues $\eta_1(t)$ and $\eta_2(t)$ slowly changing functions of time as discussed in Sec. II.

Results from this computation are shown in Figs. (6), (7), and (9) by the dashed lines. They are in good agreement with those from a computation of Eqs. (8) or (9), given in the solid lines.

The breather shapes in Fig. 7(c) at $t = 3$ agree much better than those in Fig. 7(b) at $t = 1$. This is due to a small shift in the times between the two computations evident in Fig. 6(b). At the stage $t = 1$ of breather evolution this shift produces a visible effect on $|q(x)|^2$, but at $t = 3$ where the eigenvalues are more equal and hence the period is larger the difference between the $|q(x)|^2$ is minimal.

The eigenvalue η_1 from the full computation is consistently larger than η_1 computed through the conservation laws, for equal η_2. We attribute this difference to second-order shape changes of the breather in the full equation. These will tend to diminish the change in time of especially I_3 which, in turn, is mostly reflected in a smaller change of η_2. Therefore, η_2 in the full computation lags behind the corresponding value from the conservation laws, in which second-order shape changes are excluded.

Our arguments here are patterned after those for single solitons in Sec. IV of Ref. 10, but for obvious reasons we do not attempt any quantitative analysis.

V. CONCLUSION

The good agreement between the two sets of computations demonstrates the correctness of our soliton perturbation theory, at least for the breather with two eigenvalues and with damping as perturbation. We recall that our approach is slightly restricted by the exclusion of radiation, and by the lack of intersoliton distance in the conservation laws. Our approach does have the essential feature of all soliton perturbation theories, namely, the two time-scale assumption. However, there is no particular reason besides numerical convenience and our familiarity with conservation laws to prefer their use over other approaches, nor is there anything special about breathers (again, except for convenience as noted earlier). In contrast to our method, in existing perturbation theories[14-16] damping is not singled out as a particularly suitable perturbation. Thus, it seems that the two-time-scale assumption that we have verified for damping will be valid for more general perturbations; such perturbations, then would not destroy the existence or even change the value of the eigenvalues, but they may affect the soliton shape. An example is an extra term[9] $|q|^4 q$ in Eq. (1).

Hence, we conclude that soliton perturbation theories, although justified, do not seem practical at present for anything but single solitons. Even our implementations of the two numerical methods compared in this paper used comparable amounts of computer time (5–10 sec on a CDC 7600, for the same time step $\Delta t = 0.005$ and number of grid points 128). Much additional work will be needed to develop additional approximations that increase the usefulness of soliton perturbation theories for multisolitons.

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