Energy loss of a modified Korteweg–de Vries solitary wave in a varying medium

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The energy loss from a solitary wave governed by the modified Korteweg–de Vries equation with a slowly varying coefficient is found numerically to be in good agreement with recent perturbation theory.

The modified Korteweg–deVries equation arises in the study of various weakly nonlinear dispersive systems, e.g., acoustic waves in anharmonic lattices,[1] Alfvén waves in a plasma,[2] and internal waves between two shallow layers of fluids with different density.[3] In many cases of physical interest, these waves travel in a medium which is nonuniform. When the nonuniformity is weak, these waves are governed by the modified Korteweg–de Vries equation with slowly varying coefficients.[4] A solitary wave solution of this equation based on perturbation theory through first order is given in Ref. 5. As is typical of other weakly perturbed nonlinear equations, to lowest order the solution has the form of a soliton whose parameters (amplitude, velocity, and width) vary slowly because of perturbation. However, higher-order terms in the solution often yield interesting effects, e.g., in the damped nonlinear Schrödinger equation the adiabatic damping rate is modified to second order in the perturbation strength.[6,7] In both the Korteweg–de Vries and modified Korteweg–de Vries equations with slowly varying coefficients, the first-order terms produce an irreversible energy loss from the main body of the solitary wave in spite of the absence of damping terms in the equations.[5,8] This effect is due to a transfer of energy from the main body to the trailing structure. In this note we verify the energy loss predicted by the perturbation theory for the modified Korteweg–de Vries equation with varying coefficients[5] by a comparison with the results of direct numerical integration of the differential equation.

The modified Korteweg–de Vries equation with varying coefficients can be written in the form[5]

$$u_t + \sigma(t) u^2 u_x + u_{xxx} = 0\,, \tag{1}$$

where the coefficient $\sigma(t)$ is assumed to vary slowly compared with the time scale on which the soliton varies. Initially ($t=0$), it is assumed that $\sigma=\sigma_0$ and the disturbance is a soliton given by

$$u(x,0) = a_0 \operatorname{sech}(a_0 \sigma_0^{1/2} x / 6^{1/2})\,. \tag{2}$$

Perturbation theory through first order predicts that the energy in the main body of the solitary wave is[5]

$$E(t) = E(0)[1 - \Delta(t)/2]\,, \tag{3}$$

where the initial energy is

$$E(0) = \int_{-\infty}^{\infty} u^2(x,0)\,dx = 2^{3/2} 3^{1/2} \frac{a_0}{\sigma_0^{1/2}}\,, \tag{4}$$

and

$$\Delta(t) = \frac{3^{3/2}\pi^2 \sigma_0^{3/2}}{2^{1/2} a_0^3} \int_0^t \frac{(\sigma_t)^2}{\sigma^5}\,dt\,. \tag{5}$$

In order to confirm this energy loss, numerical solutions of Eq. (1) were obtained for σ given by

$$\sigma(t) = \sigma_0 + \tfrac{1}{2}(\sigma_1 - \sigma_0)\{1 + \tanh[(t - t_0)/T]\}\,, \tag{6}$$

where $t_0 \gg T$ in order that $\sigma = \sigma_0$ at $t=0$. The quantity σ given by Eq. (6) varies smoothly from σ_0 at $t=0$ to σ_1 at $t=\infty$ with a characteristic time scale T. During the time that σ varies significantly, the numerical results show that the initial soliton given by Eq. (2) develops a trailing structure with a distinct shelf behind the main peak, and that this trailing structure changes shape and separates from the main peak after σ no longer varies. For large t(when $\sigma \approx \sigma_1$), a typical numerical solution is shown in Fig. 1 where the trailing structure has essentially separated from the main body. In this final state, Eq. (3) predicts that the main body has lost the energy

$$E(0) - E(\infty) = 6^{1/2} a_0 \Delta(\infty)/\sigma_0^{1/2}\,, \tag{7}$$

which, because the total energy is a conserved quantity for Eq. (1), has been deposited in the trailing structure. Calculating $\Delta(\infty)$ from Eqs. (5) and (6) and inserting the result into Eq. (7), we obtain

$$E(0) - E(\infty) = \frac{3\pi^2(\sigma_0+\sigma_1)(\sigma_0-\sigma_1)^2}{2 a_0^2 \sigma_1^3 \sigma_0^2 T}\,. \tag{8}$$

Figure 2 shows the main body energy loss at large time determined numerically by integrating u^2 over the trailing structure. Figure 2 is for different values of the final value σ_1 of the coefficient σ, but for constant values for the other parameters σ_0, T, and a_0. Also shown is the corresponding perturbative result from

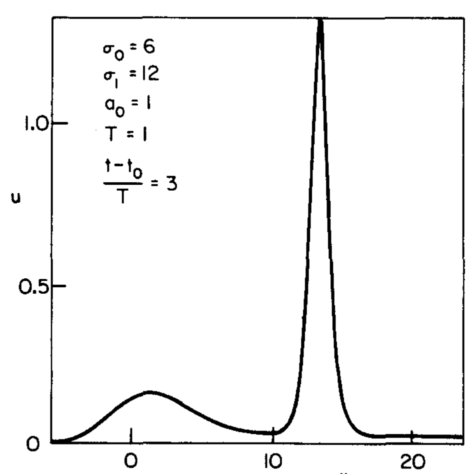

FIG. 1. Soliton and trailing structure generated by the time changing coefficient $\sigma(t)$.

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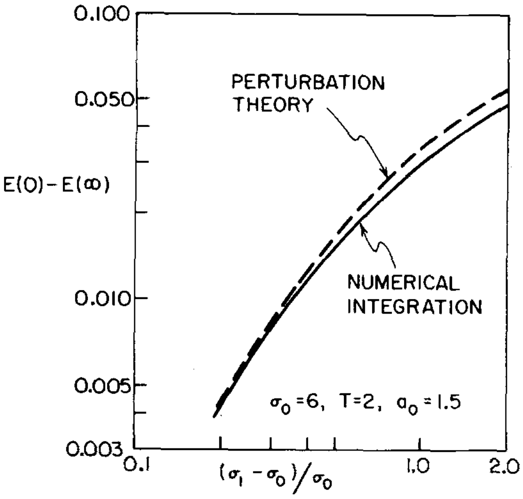

FIG. 2. Final energy loss of a soliton vs the fractional change in the coefficient σ.

Eq. (8). It is clear that the two results are in good agreement, with better agreement occurring for smaller values of the fractional change of the coefficient $(\sigma_1 - \sigma_0)/\sigma_0$. This is to be expected because Eq. (8) is based on perturbation theory which requires that[5] $(\sigma_1 - \sigma_0)/a_0^3\sigma_0^{5/2}T \ll 1$.

Figure 3 shows the influence of the time scale T on the energy loss with all other parameters constant. For sufficiently slow variation in σ, i.e., large T, the T^{-1} dependence predicted by Eq. (8) is approached. However, for small T, i.e., a rapidly varying σ, the perturbation result Eq. (8) no longer applies and Fig. 3 shows that the energy loss approaches a maximum value. This maximum energy loss occurs for an abrupt change in σ and can be found exactly by inverse scattering techniques similar to those used by Tappert and Zabusky[9] for the Korteweg–de Vries equation. Thus, by solving Eq. (1) with $\sigma = \sigma_1$ and with the initial condition given by Eq. (2), one obtains the inverse scattering result[10]

$$E(0) - E(\infty) = \frac{2^{3/2}3^{1/2}a_0}{\sigma_0^{1/2}}\left[1 - \left(\frac{\sigma_0}{\sigma_1}\right)^{1/2}\right]^2, \qquad (9)$$

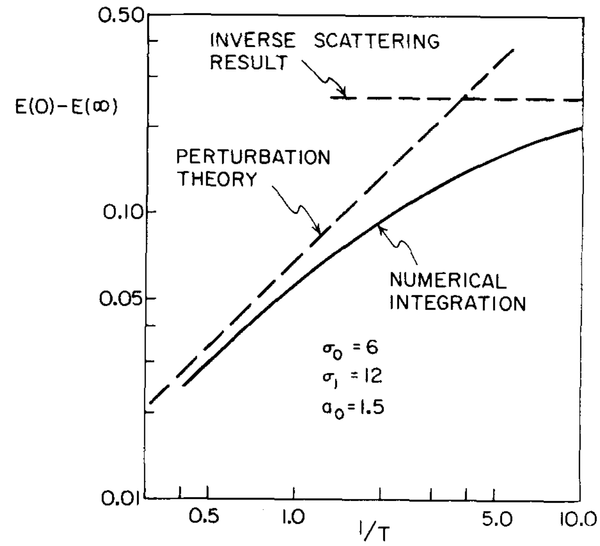

FIG. 3. Final energy loss of a soliton vs the reciprocal of the time scale T.

which is the upper limit on the energy loss as shown in Fig. 3.

In conclusion, we have numerically verified the predicted energy loss from a soliton governed by the modified Korteweg–de Vries equation with a time varying coefficient.

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