Radiation damping of long, finite-amplitude internal waves

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Numerical solutions of a damped, nonlinear wave equation are presented. The equation describes the propagation of waves in a narrow thermocline or inversion which lose energy by exciting internal waves in the weakly stratified ambient environment. The results provide estimates for the persistence of finite-amplitude internal waves propagating in a thermoclinic waveguide.

We discuss the decay characteristics of localized solutions to the nonlinear evolution equation

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}-\frac{\partial^2}{\partial x^2}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(y,t)}{x-y}dy\right)$$
$$=-\frac{\epsilon}{\pi}\int_{-\infty}^{\infty}u(y,t)dy\int_0^{\alpha}k(\alpha^2-k^2)^{1/2}\cos[k(x-y)]dk\,. \quad (1)$$

This equation was recently derived by Maslowe and Redekopp[1] in the context of finite amplitude, long internal waves propagating in a thermoclinic waveguide with vertical scale h, when the ambient fluid surrounding the thermocline is weakly stratified. They show that the wave amplitude a must satisfy the criterion

$$a/h \gtrsim O(N_\infty/N_0), \quad (2)$$

where N_0 and N_∞ denote typical Brunt–Väisälä frequencies in the main thermocline and the ambient environment, respectively. Otherwise, long waves cannot be trapped within the thermocline waveguide, but they disperse throughout the entire fluid column, and Eq. (1) does not apply. It is important to point out that Eq. (2) is a finite amplitude result; the ducting of linear internal waves in the long-wave limit is possible only when N_∞ vanishes.

The term on the right-hand side of Eq. (1) describes the damping of a trapped wave mode by the excitation and radiation of internal waves in the ambient medium. It is evident that this mechanism effectively damps the low wavenumber portion of the spectrum u_k. The cutoff wavenumber, denoted by α, is directly related to the amplitude condition given in Eq. (2), and vanishes as N_∞ tends to zero. In this limit, Eq. (1) reduces to an equation obtained by Benjamin[2] and studied intensively in recent years,[3-6]. For the physical situation described here, the equation is obtained with α of order unity, and $\epsilon=1$. The effect of α can be removed by a scaling transformation, but we choose to retain the two parameters α and ϵ in Eq. (1).

The homogeneous equation ($\epsilon=0$) possesses an infinite number of conserved densities.[7] When $\epsilon\neq0$, there is only one conserved density, namely,

$$\langle u(x,t)\rangle=\text{const}\,, \quad (3)$$

where $\langle\dots\rangle$ denotes the integral over one wavelength if the motion is periodic, or over the infinite domain if u describes a localized wavepacket. This condition requires the amplitude and length scale to vary in inverse proportion so that the wave "volume" is conserved. The next integral moment

$$\frac{\partial\langle u^2\rangle}{\partial t}=-\frac{2\epsilon}{\pi}\int_0^{\alpha}k(\alpha^2-k^2)^{1/2}|u_k(t)|^2\,dk\,, \quad (4)$$

provides a useful relation describing the energy decay of any solution to Eq. (1), where $u_k(t)$ is the spatial Fourier transform of $u(x,t)$. Maslowe and Redekopp used Eq. (4) to estimate the lifetime of solitary wave solutions

$$u(x,t)=\frac{2V/3}{V^2(x-Vt)^2+1} \quad (5)$$

of the homogeneous equation. They treat the right-hand side of Eq. (1) as a small perturbation, which is expected to give reasonable results only when $\epsilon\ll1$; that is, when the time scale for the decay is long compared with the time required for the wave to propagate one wavelength.

In what follows we present numerical solutions of Eq. (1) with the initial condition (5), in order to establish the actual damping law. The results are compared to the adiabatic theory in which u_k is evaluated from Eq. (5), treating V as a time-dependent parameter. Substituting u_k in Eq. (4) yields the relation

$$\frac{\partial\langle u^2\rangle}{\partial t}=\frac{2\pi}{9}\frac{dV}{dt}=-\frac{8}{9}\epsilon\pi\alpha^3\int_0^1k(1-k^2)^{1/2}\exp\left(-\frac{2\alpha k}{V}\right)dk\,. \quad (6)$$

A sample numerical computation is presented in Fig. 1 for the parameter choices $\alpha=2.0$ and $\epsilon=1.0$. The initial condition at $t=0$ in physical and Fourier space is displayed in Fig. 1(a), while Fig. 1(b) shows the waveform and the spectrum at $t=0.5$. At this time the wave amplitude has already decreased by 40%.

The dominant oscillation in Fig. 1(b) arises because the low wavenumbers are damped more rapidly than the higher modes. Hence, the solitary wave shape (5) can no longer be maintained in the case of strong damping, and the solitary wave develops gentle oscillations, even on the face of the wave. The wavelength of the oscillations is directly related to the cutoff wavenumber α which determines the local minimum in the spectrum. As α increases, the oscillation wavelength increases and even causes the central wave peak to exhibit a wiggly crest. On the other hand, as ϵ decreases the local minimum in the spectrum becomes less pronounced and the equilibrium solitary wave shape is more nearly maintained.

The decay of the energy as a function of the cutoff wavenumber α is presented in Fig. 2, together with the

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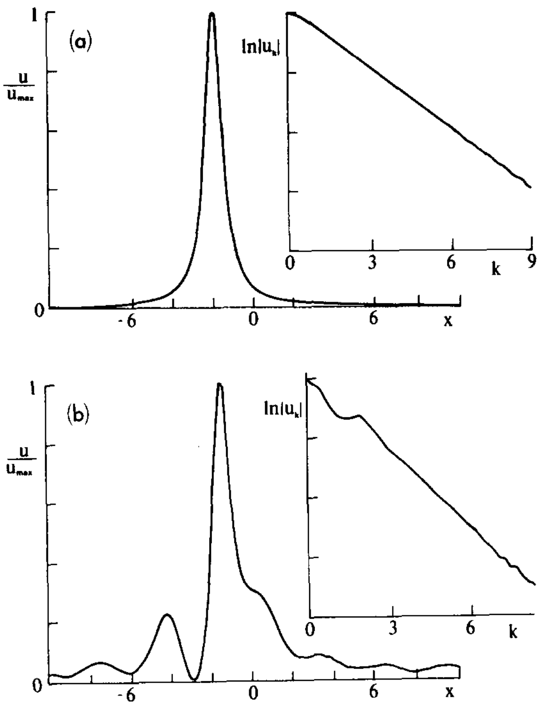

FIG. 1. The evolution of a solitary waveform and its spectrum: (a) the initial condition at $t=0$ with $V(t=0)=2.0$; (b) the evolving wave at $t=0.5$.

approximate decay obtained from Eq. (6). During the early stages the peak amplitude decreases linearly with time, but in the later stages the decay goes at t^{-1}. It is interesting that the adiabatic approximation overestimates the damping. The same result is found for the damped cubic nonlinear Schrödinger equation,[8] albeit with a quite different damping term.

The tendency for the adiabatic approximation to overestimate the damping can be understood as follows. The adiabatic theory assumes that the spectral shape [a straight line in Fig. 1(a)] is unaffected by the damping, but that the slope continually decreases with time. The true spectrum, however, is deficient in the low-

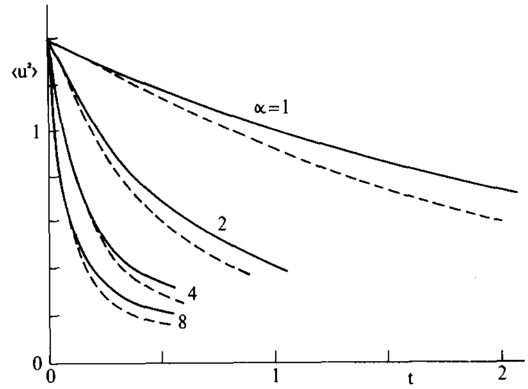

FIG. 2. The energy decay for solitary wave initial conditions: —— numerical results; ----- approximate results from Eq. (6).

wavenumber regime; hence, the lower decay rate.

The dependence of the energy decay on the parameter ϵ can be accounted for by defining a new time scale ϵt. In terms of this variable the actual decay curve approaches the adiabatic result monotonically, for given α, as ϵ decreases from unity to zero.

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