

Soliton in the damped nonlinear Schrödinger equation

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A numerical solution of the damped nonlinear Schrödinger equation is compared to analytical predictions that assume invariance of the soliton shape. The agreement is fair for the damping laws of the form $\gamma_k \propto |k|^b$. Good agreement is found for $\gamma_k \propto k^2$, and this case is studied analytically including second order effects of the damping.

I. INTRODUCTION

The nonlinear Schrödinger equation, Eq. (1), arises as the envelope equation of a dispersive wave system which is almost monochromatic and weakly nonlinear.¹ For example, two plasma heating problems of current interest in their nonlinear stage are approximated² by this equation, viz., (i) Langmuir turbulence when the background plasma is assumed³ in equilibrium with the ponderomotive pressure from the high-frequency fields, and (ii) a nonlinear stage of the mode-converted wave in the lower hybrid heating of large tokamaks.⁴

When such a wave heats (transfers energy to) the particles of the plasma, a dissipation term appears in the nonlinear Schrödinger equation. Since the heating is slow, the dissipation term is small, and can be considered as a correction that leaves some qualitative properties of the solution unchanged. In Langmuir turbulence, for instance, the dissipation is wavenumber-dependent Landau damping,^{2,3} while for the lower hybrid wave the damping is more difficult to obtain.⁵

The nonlinear Schrödinger equation is one of a class of exactly solvable evolution equations. These equations have various properties in common, notably stable nonlinear wave solutions called solitons, and an infinite set of conservation laws.⁶⁻⁸ It is well known⁸ that a large enough initial condition in such an equation typically evolves into solitons. Thus, it is of special interest to study the effect of damping on solitons.

One usually proceeds by assuming that the parameters of the soliton slowly change in time, but that the shape of the soliton, which is a reflection of the balance between nonlinearity and dispersion, does not change. This method was used earlier on the Korteweg-de Vries equation,⁹⁻¹¹ and can be justified by a two-time scale expansion.^{10,12}

In this paper we compare numerical solutions of the damped nonlinear Schrödinger equation with analytical predictions based on the first few conservation laws, in order to see whether this approximate analytical method is justified. The properties of damped solitons are derived in Sec. II. A comparison with numerical results is given in Sec. III. In Sec. IV we show that an exact solution is possible when the damping is proportional to k^2 (k is the wavenumber). Numerical results for the damped soliton were given earlier,¹³ but without comparison to an analytical theory. Damping has also been included in the inverse scattering solution of the non-

linear Schrödinger equation.¹⁴ However, that approach is rather unwieldy, and seems to yield the same results as the approximate method used here.

II. DAMPED SOLITONS: THEORY

The nonlinear Schrödinger equation for the complex function $q(x, t)$ is

$$i \frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + |q|^2 q = 0. \quad (1)$$

All variables in (1) are appropriately normalized. The damping is conveniently introduced¹⁰⁻¹² in the Fourier transformed version of Eq. (1), by giving each Fourier mode q_k , defined by $q_k = (1/2\pi) \int dx e^{-ikx} q(x)$, a damping decrement γ_k . Equation (1) then reads

$$i \frac{\partial}{\partial t} q_k - k^2 q_k + (|q|^2 q)_k + i \gamma_k q_k = 0. \quad (2)$$

Introduction of damping in this way yields an exponential decay with damping rate of γ_k for each mode when the nonlinear term $(|q|^2 q)_k$ is absent. However, the nonlinear term couples the modes, so that mode k is also affected by the damping of all other modes. The damping term $\gamma_k q_k$ in Eq. (2) has its counterpart in an extra term in Eq. (1), which becomes

$$i \frac{\partial}{\partial t} q + \frac{\partial^2}{\partial x^2} q + |q|^2 q + i \int dk e^{ikx} \gamma_k q_k = 0. \quad (3)$$

The damping term can also be written as a convolution.^{9,10}

The particular form of the damping rate γ_k depends on the physical situation where Eq. (2) applies. We mention:

- (i) Collisional damping, where the damping does not depend on wave number; $\gamma_k = \text{const} = \epsilon$, and ϵ is a small positive number.
- (ii) Landau damping, on a Maxwellian electron distribution, of Langmuir waves. In the dimensionless units of Ref. 2,

$$\gamma_k = (27\pi/2)^{1/2} k^{-3} \exp(-\frac{3}{2}k^2 - \frac{3}{2}). \quad (4a)$$

We do not use this damping in our computations because the form (4a) is too complicated for analytical results, and moreover not valid for large k . Instead, we use power-law damping, of the form

$$\gamma_k = \epsilon |k|^b, \quad (4b)$$

where the exponent b is real and nonnegative, but not

necessarily an integer. Such a damping arises in the Korteweg-de Vries equation for ion acoustic waves, where $b=1$, and for water waves with damping in a boundary layer,¹¹ where $b=0.5$. We obtain Eq. (4b) for Langmuir waves when the electron distribution has a tail proportional to v^{-b+1} . Here, we do not refer to any particular physical situation, but take Eq. (4b) as our model damping; this damping allows an analytic estimate of the properties of a soliton for large times.

For exponent b an even integer, the damping term in (3) reduces to $i\epsilon(-)^{b/2}(\partial/\partial x)^b$, so that for $b=2$ the damping term can be inserted in Eq. (3) by changing the coefficient of $(\partial/\partial x)^2$ from unity to $(1-i\epsilon)$. Collisional damping is obtained for $b=0$. For $b \neq 0$, the higher wave numbers are damped more heavily than the lower ones.

The invariance of (1) under the scaling transformation¹¹ $x' = ax$, $q(x) = aq'(x')$ (where a is an arbitrary constant), shows that a solution of (1), without damping, can be written as

$$q(x, t) = \sqrt{2}K \exp[i\theta(x, t)]f(Kx, t). \quad (5a)$$

In particular, the soliton solution is

$$q(x, t) = \sqrt{2} \frac{K \exp[i\theta(x, t)]}{\cosh[K(x - vt)]}. \quad (5b)$$

Here, K is a parameter proportional to the amplitude and inverse width of the soliton. In Eq. (5b), v is the velocity, independent of K , and $\theta(x, t)$ is the phase, given by $\theta = vx/2 - (v^2/4 - K^2)t$. The Fourier transform of the soliton (5b) (Ref. 15, Sec. 2.985) is

$$q_k(t) = \frac{\exp[i\phi(k, t)]}{\sqrt{2} \cosh[\pi(k - v/2)/(2K)]}, \quad (5c)$$

where only the phase $\phi(k, t) = (v^2/4 + K^2 - kv)t$ is time dependent.

The first three invariants of the nonlinear Schrödinger equation are

$$N \equiv \int |q|^2 dx \equiv 2\pi \int |q_k|^2 dk, \quad (6a)$$

$$P \equiv \int \left(q \frac{\partial q^*}{\partial x} - q^* \frac{\partial q}{\partial x} \right) dx \equiv 2(2\pi) \int dk k |q_k|^2, \quad (6b)$$

$$I \equiv \int \left(\left| \frac{\partial q}{\partial x} \right|^2 - \frac{1}{2} |q|^4 \right) dx. \quad (6c)$$

Here and throughout the paper the integrations run over the entire real axis. When $q(x, t)$ is interpreted as a particle probability amplitude, as in quantum mechanics, the invariant N expresses conservation of particles, P conservation of momentum, proportional to k , and I conservation of kinetic energy, proportional to k^2 , plus potential energy, proportional to $-|q|^2$. The values of the invariants for the soliton, Eq. (5b), are $N=4K$, $P=Nv$, $I = -(v^2/4 - K^2/3)N$.

When damping is introduced, N , P , and I are no longer invariant, but change in time. We find, for N

$$-\dot{N} = 2\pi \int dk 2\gamma_k |q_k|^2, \quad (6a')$$

where the dot denotes the time derivative d/dt ; and

$$-\dot{P} = 2\pi \int dk 2\gamma_k |q_k|^2 (2k), \quad (6b')$$

TABLE I. Damping integral $B_b(v')$ [Eq. (7c)] for power law damping (Ref. 15). The function $\zeta(b)$ is the Riemann zeta function.

| |
|---|
| $B_0(v') = 2$ |
| $B_1(v') = (4 \ln 2 / \pi) + v'^2/4 + \dots$ |
| $B_2(v') = 2/3 + v'^2/2.$ |
| $B_3(0) = 0.69783$ |
| $B_4(0) = 14/15$ |
| $B_b(0) = [4(b+1)/\pi^b](1-2^{1-b})\zeta(b).$ |

$$-\dot{I} = 2\pi \int dk \{ 2\gamma_k [k^2 |q_k|^2 - \text{Re}(|q|^2 q_k^* q_k)] \}. \quad (6c')$$

Equation (6a') can be interpreted as the removal of quanta; then, Eqs. (6b'), and (6c') give the change in total momentum and energy corresponding to that quantum. Even though N , P , and I are no longer invariant when damping is introduced, we will refer to Eqs. (6) as invariants or conservation laws.

How does the soliton of Eq. (5b) change on introduction of damping? First consider the solution (5a), with velocity $v \equiv \partial\theta/\partial z = 0$. If $f(Kx, t)$ corresponds to a solution of Eq. (1), it is reasonable to assume that only the parameter K depends to order ϵ on time, but that the shape $f(Kx, t)$, expressing an equilibrium between dispersion and nonlinearity, depends to some higher order on time; we write $f = f(Kx; t)$.

The time change of K must be consistent with the conservation laws, in particular with Eq. (6a'). Calculating the Fourier transform $f_k(t)$ of (5a), introducing $k' = k/K$, and substituting this in (6a') we find

$$(d/dt)(K^{-b}) = \epsilon b B_b(f), \quad (7a)$$

where

$$B_b(f) \equiv \pi \int dk' |k'|^b |f_{k'}(t)|^2. \quad (7b)$$

For fixed b , the coefficient $B_b(f)$ is a functional of the Fourier transform $f_{k'}(t)$, or equivalently of the shape of the solution $f(Kx; t)$. If this shape does not change, B is constant in time. Conversely, when B changes in time we know that the shape of the soliton changes (as found in Sec. III).

Not only the shape, but also the initial velocity influences the coefficient B . Introducing the Fourier transform (5c) of a soliton with nonzero initial velocity v in Eq. (6a') yields Eq. (7a), with $B_b(f)$ replaced by

$$B_b(v') = \frac{\pi}{2} \int \frac{dk' |k'|^b}{\cosh^2[(k' - v'/2)\pi/2]}. \quad (7c)$$

Here, $v'(t) = v(t)/K(t)$ can change in time, since K and v depend on time. As both $|k|^b$ and $\cosh^2(k\pi/2)$ are even in k , $B_b(v')$ is a quadratic function of v' for small v' . Consequently, $B_b(v')$ changes little in time when the initial velocity is small enough, since only the terms of B quadratic in v' change in time. Values of $B_b(v')$ for various b and v' are given in Table I.

When B is constant, we can integrate Eq. (7a) to find

$$K(t) = K_0 / (1 + \epsilon vt)^{1/b}, \quad (8a)$$

with $K_0 \equiv K(0)$, and the damping coefficient ν

$$\nu = bK_0^b B_0. \quad (8b)$$

In the limit of collisional damping, $b \rightarrow 0$, we recover exponential decay, $K(t) - K_0 \exp(-\epsilon B_0 t)$. The form of the time dependence in Eq. (8a) is the same as found¹¹ for this quantity in the Korteweg-de Vries equation, but the coefficient, which depends on the functional form, is different. Moreover, we have the free parameter ν' , which is absent in the Korteweg-de Vries equation. The damping coefficient of Eq. (8b) is compared to the computational result in Sec. III.

For Landau damping the integral (6a') can be evaluated approximately² with a saddle point method. For our purposes, however, it is sufficient to estimate $\gamma_k = 0$ for $k < k_0$ and $\gamma_k = c$, for $k > k_0$, where k_0 and c are constants of order unity. A soliton with initially large amplitude and inverse width K is damped strongly as long as $K > \pi k_0$, but when K falls below this value the damping disappears rapidly. (Landau damping is compared to power law damping in Fig. 6.)

Having found the time dependence of the soliton amplitude and inverse width K , we now turn to the effect of damping on the velocity. It is clear from the symmetry of Eq. (3) for space reflections $x \rightarrow -x$ that a soliton initially at rest, $v(0) = 0$, stays at rest throughout the damping. A soliton with initially positive velocity, however, decreases its speed if the damping increases with wavenumber. This is because the velocity determines the position $k_m = v/2$ in k space of the maximum of the spectrum $|q_k|^2$. The spectrum itself is symmetric around k_m . Since the damping is larger for values $k_m + |k'|$ than for $k_m - |k'|$, where $|k'|$ is the distance to k_m , more of the spectrum disappears at larger wavenumbers; this shifts the maximum k_m to lower wavenumbers, and consequently, the velocity decreases.

The time dependence of the velocity can be estimated from Eq. (6b') for \dot{P} . We rewrite this equation by putting $k = (k - v/2) + v/2$, and use $\dot{P} = v\dot{N} + N\dot{v}$ and Eq. (6a) to find

$$N\dot{v} = -8\pi \int dk \gamma_k (k - v/2) |q_k|^2. \quad (9)$$

The function $(k - v/2)|q_k|^2$ is antisymmetric in its argument, and the integral vanishes when γ_k is independent of wavenumber, i.e., for collisional damping where $b = 0$.

For wavelength-dependent damping, $b \neq 0$, the absolute values in $\gamma_k = \epsilon |k|^{1/b}$ preclude the convenient calculation of Eq. (9) for arbitrary b . However, when b is even, the absolute values in γ_k are immaterial. For $b = 2$, for example, we again split k in $(k - v/2)$ and $v/2$ and insert this in $\gamma_k = \epsilon k^2$. The integration in Eq. (9) annihilates the terms of γ_k even in $(k - v/2)$, and we obtain

$$N\dot{v} = -8\pi \epsilon v \int dk (k - v/2)^2 |q_k|^2. \quad (10)$$

When the initial velocity is small, we can neglect v in the integral, and compare Eqs. (10) and (6), which leads to

$$N\dot{v} = 2v\dot{N}, \quad (11)$$

whence

$$v(t) = v_0 [N(t)/N_0]^2, \quad (12)$$

[v_0 denotes the initial velocity, N_0 is the initial value $N(0)$.]

The position of the soliton maximum x_m , which can easily be compared to the numerical results, follows by integrating Eq. (12) once more. Using the explicit time dependence of Eq. (8) yields

$$x_m(t) = (v_0/\epsilon v) \ln(1 + \epsilon vt). \quad (13)$$

It remains to be seen whether the third conservation law, Eq. (6c') is satisfied for our approximate solution. This is, of course, not sufficient to make the approximate solution an exact one (for an exact solution we must use all conservation laws, not only three, or go back to the original equation), but makes it more likely that the approximate solution is reasonable. The Fourier transform of $|q|^2 q$ is

$$(|q|^2 q)_k = (k^2 + K^2 + v^2/4 - kv) q_k \quad (14)$$

(Ref. 15, Sec. 3.985.3). Inserting this in Eq. (6c') cancels the first term. Using $\dot{I} = -4K^2 \dot{K} + v^2 \cdot \dot{K} + 2vK\dot{v}$ cancels the K^2 term from Eq. (14) when Eq. (6a') for $4\dot{K}$ is invoked. Transposing $v^2 \dot{K}$ to the other side, using Eq. (6a') once more, and combining terms finally yields $2v$ times Eq. (9b). Thus, the first three conservation laws, Eq. (6'), can be satisfied by suitable choices of $K(t)$ and $v(t)$ in the soliton functional form of Eq. (5). We also note that the integral $\int |q| dx$, which is proportional to the number of solitons⁸ that can evolve from a particular initial condition $q(x, t=0)$, is independent of K for a soliton, Eq. (5); thus, the number of solitons does not change when damping is included.

Throughout this discussion we have tacitly assumed that ϵ is positive, i.e., damping rather than growth. However, Eq. (3) is invariant under the complex conjugation when in addition t and ϵ change sign. Thus, our results are also valid for a wavelength-dependent growth, as long as the use of Eqs. (5) and (6) remains justified.

III. COMPARISON TO NUMERICAL SOLUTION

We now compare the time evolution of a soliton with power-law damping in Eq. (3) to the analytical prediction of scale invariance, viz., (i) decrease of N as a power law with exponent $1/b$, Eq. (8a), and (ii) sech-shaped soliton functional form, or numerical value of the damping coefficient ν [Eq. (8b), with B_0 from Eq. (7b)]. The numerical results were obtained by solving the Fourier transformed Eq. (3). A system of length L is considered with discrete wavenumbers $k = 2\pi m/L$ where m is an integer. The linear terms on the left of this equation are integrated exactly and increments due to the nonlinear terms are applied using an implicit method with iterations. Modes corresponding to $-m_{\max} \leq m \leq m_{\max}$ are retained in the computations. The convolution sums required in the nonlinear terms are computed by transforming back to x space after the addition

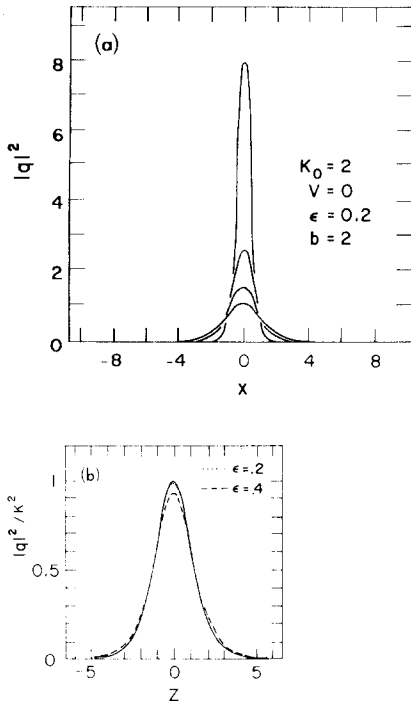


FIG. 1. (a) Evolution of a damped soliton. Shown is $|q(x,t)|^2$ versus x for times $t=0, 2, 4,$ and 6 . Initial soliton parameters are $K=2, v=0$; damping parameters are $\epsilon=0.2,$ and $b=2$. (b) Stationary shape of the damped soliton. Shows is $|q(z)|^2/K^2,$ with $z=K(t)x,$ for damping exponent $b=2$. The solid line is the initial sech shape. The dotted line shows the same data as the previous figure for $t=2, 4,$ and $6,$ on the same curve. The dashed line is the shape for larger damping $\epsilon=0.4.$

of zero modes to eliminate the periodicity in k space. All computations are carried out in k space and representation in x space is used only in evaluating convolutions or for diagnostic purposes.

The initial parameter K was arbitrarily chosen as $K=2$ in all computations without loss of generality,¹¹ leaving the damping strength $\epsilon,$ the damping exponent $b,$ and the initial velocity as parameters.

Figure 1(a) shows the time evolution of a soliton with initial velocity $v=0$ subject to power-law damping with parameters $\epsilon=0.2$ and $b=2$. The soliton shape seems rather well preserved in time, and the maximum amplitude $2K^2$ decreases while the width K^{-1} increases. We see no evidence of outgoing waves but the soliton remains localized. The area under the soliton $\int |q|^2 dx = N$ decreases in time.

The functional forms of the soliton at $t=0,$ and at $t=2, 4,$ and $6,$ are compared in Fig. 1(b) by plotting $|q|^2/K^2(t)$ versus the argument $Z \equiv K(t)x.$ For small damping, $\epsilon=0.2,$ the shapes fall for all times on the dotted curve. Thus, the shape is indeed independent of time, but lower than the initial sech² form. For larger $\epsilon=0.4,$ the invariant shape, given by the dashed curve, is also wider than at $t=0.$

The time dependence of N for this same case $b=2$ is presented in Fig. 2. Referring to Eq. (7a), we show $[K(t)/K_0]^{-2}$ which should be linear in time. We display

two measures for $K,$ viz., (i) the integral $\int |q|^2 dx = 4K,$ in the solid curve, and (ii) the maximum of the soliton $|q|_m^2 = 2K^2$ (broken curve). These curves should coincide if the soliton kept its original sech shape. For small damping, $\epsilon=0.2,$ these lines indeed coincide. However, for larger damping, $\epsilon=0.4,$ the slopes of the lines are different. This difference indicates that the soliton has a lower amplitude and is slightly wider than expected from the sech form. Moreover, the decrease in N is slower at later times than at earlier times. This slower decrease is consistent with the widening of the soliton. Wider solitons in x space have narrower spectra in k space, thus lowering the integral for $B,$ Eq. (7b). The numerical value in the computation of $B,$ is slightly lower than the theoretical value $B_2(0)=2/3,$ the difference between them increasing damping strength ϵ [see Fig. 3(b)]. However, even for the rather large value $\epsilon=0.4$ for the damping strength, the concept of a soliton decaying as assumed in Sec. II is quantitatively valid to within 10%. The evolution of a damped soliton for $b=2$ is reconsidered in Sec. IV.

For other values of b the soliton form changes, and so does the damping coefficient. Figure 3 compares the instantaneous damping coefficient ν_c as a function of time [cf. Eqs. (7a) and (8b)],

$$\nu_c(t) \equiv \frac{N_0^b}{\epsilon} \left. \frac{d[N(t)^{-b}]}{dt} \right|, \quad (15)$$

to the theoretical value ν_{th} for the integer values of $b=1, 2, 3, 4.$ [The time dependence of N for collisional damping ($b=0$), of course, follows the predicted exponential decay $N(t)=N_0 \exp(-2\epsilon t)$ very well, since the decay constant 2ϵ does not depend on the functional form of the soliton.] We have chosen the damping strength ϵ as a function of the damping exponent b such that the initial change in $N,$ the right-hand side of Eq. (7a), is about the same for all $b.$ With this ϵ we have, in some sense, corresponding strength for the different values of the damping exponent.

When $b=1,$ the damping coefficient is slowly increasing from the theoretical value for small times, but de-

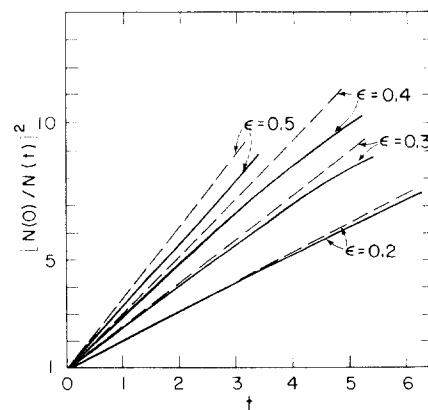


FIG. 2. Soliton scale factor $K(t)$ versus time for power-law damping with damping exponent $b=2,$ and various values of damping strength $\epsilon.$ The solid line is $[K(0)/K(t)]^2$ based on $N,$ the broken line based on $q_m^2.$ Soliton parameters are $K=2,$ and $v=0.$

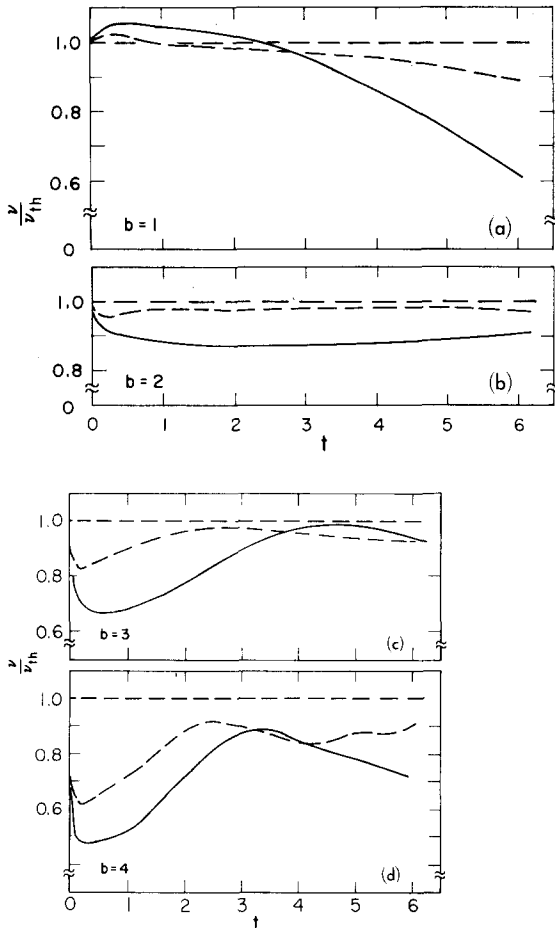


FIG. 3. The computed damping coefficient $\nu_c(t)$ [Eq. (15)] versus time for different b . The horizontal line at unity is ν_{th} from Eq. (8b). The broken curve is for the smaller ϵ , the solid curve for the larger ϵ . Parameters are: (a) $b=1$, $\epsilon=0.3$, and $\epsilon=0.6$. (b) $b=2$, $\epsilon=0.2$, and $\epsilon=0.4$. (c) $b=3$, $\epsilon=0.1$, and $\epsilon=0.2$. (d) $b=4$, $\epsilon=0.035$, and $\epsilon=0.07$.

creases later to deviate from ν_{th} up to 10% when $\epsilon=0.3$, and up to 40% when $\epsilon=0.6$, as shown in Fig. 3(a). For $b=2$, Fig. 3(b), the damping coefficient for $\epsilon=0.2$ and 0.4 is almost constant, but slightly lower than ν_{th} , as mentioned earlier. The damping coefficient for $b=3$, Fig. 3(c), decreases rapidly from ν_{th} to reach a minimum of 0.65 of ν_{th} for $\epsilon=0.2$, but then increases while oscillating. The case $b=4$, given in Fig. 3(d), reaches an even lower minimum, but otherwise behaves similarly to the previous one. When $b=3$ or $b=4$, the damping coefficient appears to reach a saturation value slightly lower than ν_{th} for later times.

How do we explain these deviations from expected behavior? First consider the case $b=2$, which corresponds fairly closely to theory. Apart from $t=0$, to which we return in a moment, ν_c stays constant, albeit smaller than the theoretical value ν_{th} . Thus, the decrease in N is given by the power law derived from similarity arguments, but the functional form of the soliton has changed slightly to account for the change in damping coefficient. It is reasonable to assume that the damping, proportional to k^2 , preferentially depletes the tail of the spectrum. Then, the damping coefficient

decreases as is clear from Eq. (6a). This damped soliton is slightly wider in x space with the same amplitude, or slightly lower, with the same width, than a pure sech-shaped soliton.

The assumption that a soliton keeps its functional form is apparently not satisfied near $t=0$. The steep decline in $\nu(t)$ for times smaller than $t=0.3$ indicates that $\int dk \gamma_k |q_k|^2$ decreases rapidly in time; indeed, the tail of the spectrum is cut off by the damping, and the nonlinearity has not had time to replenish it; thus, little spectrum is left to damp. After this initial stage the soliton for $b=2$ seems to be in a self-similar equilibrium, but for $b \neq 2$ no such equilibrium is apparent. For $b=1$, the damping coefficient ν_c is larger but fairly close to ν_{th} , and decreases as time goes on. For $b=3$ and $b=4$, the damping coefficient seems to return to the theoretical value, and at least for the smaller damping, $\epsilon=0.1$ and $\epsilon=0.35$, ν_c oscillates around an equilibrium near ν_{th} .

The shape of a damping soliton as a function of time is shown in Fig. 4(a) when the damping exponent $b=1$. At $t=2$, the soliton shape is more peaked than the initial sech-form. At $t=4$ the peak has grown, and additional tails have appeared.

The case $b=3$ is displayed in Fig. 4(b). The maximum of the soliton at $t=2$ is now lower than the initial sech shape, while this maximum at $t=4$ is in between these two values. No tails are visible. The changing shape of the soliton in the cases $b=1$ and $b=3$ contrasts

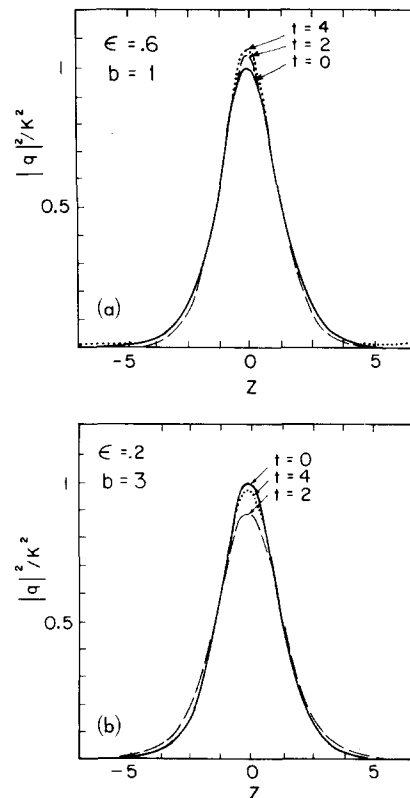


FIG. 4. Shape of the damped soliton. Shown is $|q(z;t)|^2/K^2$, with $z=K(t)x$, and $K=N/4$ from the computation. Parameters are (a) $\epsilon=0.6$, $b=1$. (b) $\epsilon=0.2$, $b=3$.

with the stationary shape of the soliton for $b = 2$, shown in Fig. 1(b), and qualitatively explains the change in time of the damping rate.

Another feature seems to be that the difference between ν_c and ν_{th} increases as time goes on for $b = 1$, the difference decreases for $b = 3$ and $b = 4$ in time, while $|\nu_c - \nu_{th}|$ remains approximately constant for $b = 2$. We explain the b dependence of $|\nu_c - \nu_{th}|$ by the properties of the damping term under the scaling¹¹ transformation $t' = a^2 t$, $x' = ax$, $q'(x', t') = a^{-1} q(x, t)$. This transformation multiplies Eq. (1) with an overall factor a^2 . However, the damping term $\gamma_k q_k$ scales as a^{b-2} relative to the other terms in Eq. (3) or (2) (as $\gamma_k = a^b \gamma'_k$, where $k' = a^{-1} k$). Thus, the relative size of the damping for $b = 1$ increases as the scaling factor a , which can be taken proportional to $N(t)$ or to $K(t)$, decreases in time. The relative size of the damping terms stays constant when $b = 2$, and decreases for b greater than two. Consequently, the influence of the damping on the form of the soliton becomes larger for $b = 1$, but diminishes for $b = 3$ and $b = 4$.

Naturally, the smaller ϵ the smaller the disagreement between analytical and numerical results. We note that the differences between ν_c and ν_{th} for $b = 2$ and $b = 3$ typically increase by a factor of four when ϵ is increased by a factor of two. We conclude that, apart from an initial transient, the soliton behaves substantially as predicted when ϵ is chosen small enough. However, "small enough" means ϵ of the order 0.01 for $b = 4$, around 0.03 for $b = 3$, while only for $b = 2$ can we have a "small enough" ϵ larger than 0.2. The given values are illustrative for a soliton with parameter $K = 2$: For solitons with other parameters, the illustrative ϵ 's change by a factor $(K/2)^{2-b}$ when a soliton with parameter K is damped.

It is of interest to determine the functional form of a soliton that remains self-similar from $t = 0$ to all later times; this is only possible for $b = 2$, since only then does the damping term not change in time relative to the other terms. We defer this discussion to Sec. IV.

The soliton in all previous computations had an initial velocity of zero. This velocity remains zero as Eq. (3)

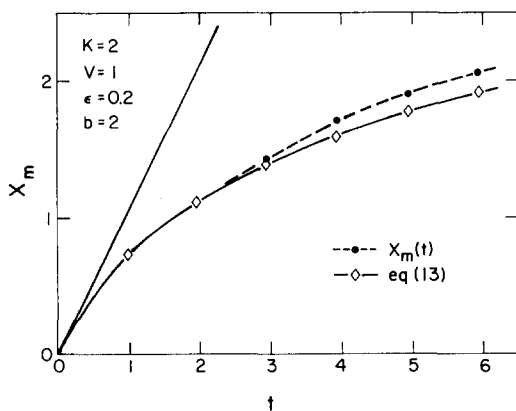


FIG. 5. Position of the maximum x_m versus time of the damped soliton with $K = 2$, $v = 1$, $\epsilon = 0.2$, and damping exponent $b = 2$. The straight line is the position of an undamped soliton with the same initial conditions.

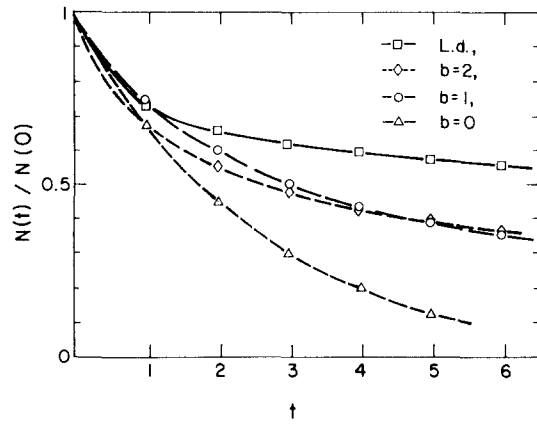


FIG. 6. N versus time for Landau damping (Ld, open squares), and power-law damping. The initial parameters are $K = 3.5$ for Landau damping, and $K = 2$, $\epsilon = 0.2$ for power-law damping. The velocity $v = 0$ in all cases.

is symmetric in x . For collisional damping the velocity does not change even when $v \neq 0$. We now briefly consider a moving soliton. In Fig. 5 we show the position x_m of the soliton maximum by the broken line when the damping exponent $b = 2$, i. e., the case that follows the analytic approximations best. The straight line is the position in the undamped case, $x_m = vt$ with $v = 1$. We observe fair agreement with the prediction of Eq. (13), given by the solid curve. When the initial velocity is not equal to zero, the damping is indeed stronger than when $v = 0$ (cf. Table I), but as v decreases rapidly in time the damping coefficient quickly regains its value for $v = 0$.

The other damping exponents show qualitatively similar behavior, but as even the damping of N for $v = 0$ does not follow our expectations very well for $b \neq 2$, we have not made a quantitative comparison between the observed and expected positions of the maximum in those cases. We just note that $x_m(t)$ for $b = 1$ is very well fitted by the empirical relation $x_m(t) = v_0 t N^{1/2}(t)$.

After developing some understanding of the analytical-somewhat tractable case of power-law damping we conclude with a soliton subject to Landau damping. In Fig. 6 the time evolution of $N(t)$ with Landau damping, the curve marked Ld, is compared to this quantity, when power-law damping with $b = 0$, $b = 2$, and $b = 4$ applies. We have again chosen parameters such that the initial damping is about the same in all computations. We see that $|N|_{Ld}$ is large initially, compared with its final value, which is almost zero. The decrease in N for larger times is more rapid with power-law damping than with Landau damping.

In this section we have seen that a soliton in the nonlinear Schrödinger equation with damping remains localized, but changes shape in the course of time. Consequently, the time development of the invariant $N(t)$, or the parameter $K(t)$, is different from the predictions, but of the correct order of magnitude. When the damping is proportional to k^2 , the functional form of the soliton remains invariant. This case is studied more closely in the next section.

IV. SELF-SIMILAR SOLITON FORM FOR $b = 2$

In this section we revert to the problem of finding a soliton that changes only the parameter $K=K(t)$ when damping is introduced, but does not change its functional form. Such a soliton can exist only for $b = 2$, because only in that case does the magnitude of the damping term, when compared to the other terms in Eq. (3), remain constant in time (see the discussion in Sec. III). For convenience, we assume that the velocity of the soliton is zero, and that its maximum is at $x = 0$.

When $b = 2$, Eq. (3) becomes

$$i \frac{\partial q}{\partial t} + (1 - i\epsilon) \frac{\partial^2 q}{\partial x^2} + |q|^2 q = 0. \quad (16)$$

We assume a solution of the form

$$q(x, t; \epsilon) = \sqrt{2}K(t; \epsilon)f(z; \epsilon) \exp[i\theta(z, t; \epsilon)]. \quad (17)$$

Here $z(x, t; \epsilon) \equiv K(t; \epsilon)x$, and f and θ are real functions of the indicated variables. In the limit $\epsilon \rightarrow 0$ the functions f and θ must reduce to the ones for an undamped soliton, $f(z; 0) = \text{sech}(z)$ and $\theta(z, t; 0) = K^2 t$ [compare Eq. (5b)]. Substitution of Eq. (17) in Eq. (16) and separation of the real and imaginary parts yields

$$-\dot{K}/K^3 z \theta_z f - \dot{\theta}/K^2 f + f_{zz} + 2f^3 - \theta_z^2 f + \epsilon(2\theta_z f_z + \theta_{zz} f) = 0, \quad (18a)$$

$$\dot{K}/K^3 (f + z f_z) + (2\theta_z f_z + \theta_{zz} f) - \epsilon f_{zz} + \epsilon \theta_z^2 f = 0, \quad (18b)$$

where the dot denotes $\partial/\partial t$ and the subscript z stands for $\partial/\partial z$. From Eq. (18b) we see that θ_z can be independent of time, if \dot{K}/K^3 is independent of time. In view of Eq. (7a), we write

$$\dot{K}/K^3 = -\epsilon B(\epsilon), \quad (19)$$

where the constant $B = B(\epsilon)$ is to be determined. When θ_z is independent of time, $\dot{\theta}_z = 0$, and it follows that $\dot{\theta}$ is independent of z .

Then, Eq. (18a) shows that $-\dot{\theta}/K^2$, the normalized nonlinear frequency shift, is constant. We choose this constant -1 , as in the undamped case. Thus, $\theta(z, t; \epsilon)$ can be written as

$$\theta(z, t; \epsilon) = \epsilon g(z; \epsilon) + \int_0^t K^2(t') dt'. \quad (20)$$

The z -dependent part of the phase must be chosen proportional to ϵ , because without damping the phase $K^2 t$ of a soliton with velocity $v = 0$ is independent of x or z . Using Eqs. (18b)–(20) we find two equations for the shape $f(z; \epsilon)$ and the spatial dependence of the phase $g(z; \epsilon)$

$$(-1 + \epsilon^2 B)f + (1 + \epsilon^2)f_{zz} + 2f^3 + \epsilon^2[Bz(g_z f + f_z) - g_z^2(1 + \epsilon^2)f] = 0, \quad (21a)$$

$$\epsilon[-B(f + z f_z) + 2g_z f_z + g_{zz} f - f_{zz} + \epsilon^2 g_z^2 f] = 0 \quad (21b)$$

without any time dependence. This nonlinear eigenvalue problem is an exact consequence of the time-dependent Eqs. (16) and (17), showing that the assumption of a stationary solution can be satisfied exactly. The solution, however, is clearly very complicated. Rather than finding a full solution we now show that a solution exists to order ϵ^2 , and calculate the damping rate $B(\epsilon)$

to this order for comparison with the numerical results shown in Fig. 3(b).

The undamped soliton following from Eq. (21) is, of course,

$$f_0 \equiv f(z; 0) = \text{sech } z. \quad (22)$$

To first order in ϵ the shape is also given by Eq. (22) because the correction terms in Eq. (21a) are of order ϵ^2 or higher. Thus, the assumption of invariant soliton shape is justified to first order in ϵ .

The damping rate is calculated by multiplying Eq. (21b) by f , integrating, and solving for B

$$B(\epsilon) = \frac{\int (2 \int f_z^2 dz + 2\epsilon^2 \int f^2 g_z^2 dz)}{\int f^2 dz}. \quad (23a)$$

Inserting $f_0 = \text{sech } z$ yields $B = 2/3$, correct to first order in ϵ . Alternatively, B can easily be found to lowest order in ϵ without the explicit solution, because Eq. (23a) only contains integrals over $f(z)$. We put $\epsilon = 0$; multiplying Eq. (21a) by $f/2$ and integration over z yields

$$\frac{1}{2} \left(\int f^2 dz + \int f_z^2 dz \right) = \int f^4 dz. \quad (23b)$$

Multiplying Eq. (21a) by $2f_z$, integrating [compare Eq. (25a) with $\epsilon = 0$] and integrating once more over z gives

$$\int f^2 dz - \int f_z^2 dz = \int f^4 dz. \quad (23c)$$

Combination of Eqs. (23a)–(c) also leads to

$$B(0) = 2/3. \quad (23d)$$

Calculation of the second order correction to the functional form and damping rate demands the first order phase g from Eq. (21b). Neglecting the ϵ^2 -term, multiplying by f_0 , integrating, and using $B = \frac{2}{3}$ yields

$$g_z = -\frac{2}{3} \tanh z + z/3, \quad (24a)$$

$$g = \frac{2}{3} \ln(\text{sech } z) + z^2/6. \quad (24b)$$

The first term of Eq. (24) can be included in f as a complex exponent. To first order, Eq. (17) becomes (compare Ref. 16)

$$q(x, t; \epsilon) = \sqrt{2}K(\text{sech } z)^{1+2i\epsilon/3} \exp(i\epsilon z^2/6 + i \int_0^t K^2 dt). \quad (24c)$$

The soliton to second order in ϵ can be found by multiplying Eq. (21a) by $2f_z$ and integrating. We find

$$(1 + \epsilon^2)f_z^2 + V(f; z) = 0, \quad (25a)$$

where

$$V(f; z) = f^4 - (1 - 2\epsilon^2/3)f^2 + \epsilon^2 \delta V(f; z), \quad (25b)$$

and

$$\delta V(f; z) = 2 \int X f_z dz + \text{const.} \quad (25c)$$

Here, $X = X(f; z)$ stands for the term in the brackets of Eq. (21a). The integration in Eq. (25c) is the indefinite integral; the constant is chosen such that $\delta V = 0$ when $f = 0$.

The indefinite integration with the unknown function $f(z)$ leads to seemingly dissipative terms such as $Bz f_z^2$ in the integrand. This would preclude writing δV as a

potential solely dependent on f as needed for the calculation of f with Eq. (25a). However, δV need only be calculated to lowest order in ϵ ; thus, we can use $f_0 = \text{sech} z$ instead of the unknown function $f(z)$ in the integration. The difference between f_0 and f is of order ϵ^2 , and because these functions go to zero fast enough as $z \rightarrow \infty$, the difference between integrals of these functions is also of order ϵ^2 [see Fig. 1(b); f_0 is the solid curve, f is the dotted or broken curve].

This procedure leads to (cf. Ref. 15, Sec. 2.447):

$$\delta V = [(-\frac{1}{2}z^2 + 2z \tanh z - \text{sech}^2 z + 3)\text{sech}^2 z - z \tanh z - \ln(\text{sech } z) + \ln 2]/9. \quad (26)$$

The constant $\ln 2$ is the limit of $z \tanh z + \ln(\text{sech } z)$; thus $\delta V \rightarrow 0$ as $f \rightarrow 0$. The desired form $\delta V(f)$ is found, correct to first order in ϵ , by substituting f for $\text{sech } z$, $(1-f^2)^{1/2}$ for $\tanh z = -f_x/f$, and $\text{arcsech}(f)$ for z .

It is now clear that $V(f; z) = V(f)$ fulfills the requirements for a soliton solution, viz. (i) at $z=0$, $f_x=0$ and $V(f)=0$, and (ii) at infinity, $f_x=0=f$, together with $V=0 = \partial V/\partial f$. Thus, a soliton-type solution exists to second order in ϵ , and its functional form can, in principle, be found by integration of Eq. (25a), with Eq. (26).

Fortunately, it is not necessary to find the explicit form of this solution $f(z)$ for the calculation of the damping rate, because Eq. (23a) for B only contains integrals over $f(z)$. Performing the steps that lead to Eq. (23d) but without putting $\epsilon=0$, using $\int f^2 dz = 2 + O(\epsilon^2)$ whenever it multiplies an $O(\epsilon^2)$ quantity, and gathering terms of order ϵ^2 gives

$$B(\epsilon) = \frac{2}{3} [1 - \epsilon^2 (\frac{5}{3} - \frac{1}{2} \int X f dz + \int \delta V dz - \frac{3}{2} \int f^2 g_x^2 dz)] + O(\epsilon^4). \quad (27a)$$

Calculation of the integrals in (27a) to lowest order in ϵ yields

$$\int X f dz = (26 - \pi^2/2)/27, \quad (27b)$$

$$\int \delta V dz = - \int z \partial \delta V / \partial z dz = (\pi^2/4 - 20)/27, \quad (27c)$$

$$\int f^2 g_x^2 dz = (\pi^2/4 - 2)/27, \quad (27d)$$

using Ref. 15, Sec. 3.527. In addition we need $I \equiv \int x^2 \times \text{sech}^4 x dx$. With $(\partial/\partial a)^2$ of $\int \text{sech}^2 ax dx = 2/a$, we find $I = \pi^2/9 - 2/3$. The final result is

$$B(\epsilon) = \frac{2}{3} [1 - \epsilon^2 (15 + \pi^2/8)/27] \approx \frac{2}{3} (1 - 0.6\epsilon^2). \quad (28)$$

Comparing Eq. (28) with the data displayed in Fig. 3(b) we find excellent agreement: for $\epsilon = 0.2$, $B(0.2)/B(0) \approx 0.976$ from Eq. (28), versus 0.98 from Fig. 3(b); for $\epsilon = 0.4$, we find $B(0.5)/B(0) \approx 0.904$ versus 0.89.

V. CONCLUSION

We have made a detailed comparison of numerical soliton solutions of the nonlinear Schrödinger equation with damping, with the analytical assumption that solitons keep their shape, while damping as predicted by the first few conservation laws.

We have investigated different damping laws, all of

the form $\gamma_k = \epsilon |k|^b$. A standing damped soliton with damping exponent $b=1$ develops small tails, but solitons with damping exponent $b \geq 2$ remain localized. For the damping exponent $b \neq 2$ the soliton form changes slightly, but this change has a sizable influence on the damping coefficient, defined by Eq. (7a). In contrast, when $b=2$ the soliton shape and the damping coefficient are constant in time. This qualitative difference between $b \neq 2$ and $b=2$ is explained with the scaling properties of the nonlinear Schrödinger equation compared to its damping term.

The velocity of a soliton for $b=2$ decreases in agreement with the second conservation law; we have given a qualitative explanation of this decrease, valid for $b \geq 0$.

For the damping exponent $b=2$ we have found a formula for the stationary shape of the damped soliton, correct to second order in ϵ , and calculated the damping rate to this order. This damping rate shows excellent agreement with the numerical solution.

We add a note of caution: These results cannot be taken over automatically in other nonlinear wave equations. For instance, solitons in the Korteweg-de Vries equation do not behave as nicely as the solitons considered here.¹⁷

In the Korteweg-de Vries equation, $\partial q/\partial t + q\partial q/\partial x + \partial^3 q/\partial x^3 = 0$, the integral $\int q dz$ is conserved in addition to Eq. (6a'); the damping soliton decreases amplitude, velocity, and inverse width according to Eq. (6a'), but also leaves a wake which accounts for the conservation of $\int q dx$. The wake has a small amplitude, and behaves almost linearly, i.e., disperses into the negative direction.

Initial conditions of other than stationary soliton form¹⁸ in the nonlinear Schrödinger equation behave in a more complicated manner, but we have not systematically investigated these.

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