fact that the Child–Langmuir law is obeyed supports the validity of our method and is consistent with turbulence theory[1] which suggests that anomalous behavior occurs only when the drift velocity exceeds the thermal velocity 1.3 times.

These results suggest that the resonance-cone technique is a powerful diagnostic tool for measuring both the electron temperature and drift velocity when the electron-plasma frequency is known and is much larger than the electron gyrofrequency.

In future experiments we intend to check the validity of this technique in the turbulent regime when $V_d > 1.3\ V_t$.

We would like to thank Dr. L. R. O. Storey for many useful discussions and for the original idea from which this study grew.

[a)]Present address: Plasma Research Laboratory, Research School of Physical Sciences, The Australian National University, Canberra, Australia.

[1]W. H. M. Clark and S. M. Hamberger, Plasma Phys. 21, 943 (1979).
[2]J. P. Lebreton and D. Henry, Phys. Lett. A 75A, 49 (1980).
[3]L. R. O. Storey and R. Pottelette, C. R. Acad. Sci. 273, 101 (1971).
[4]J. M. Illiano and R. Pottelette, Phys. Lett. A 70A, 315 (1979).
[5]K. Lucks and M. Krämer, Plasma Phys. 22, 879 (1980).
[6]A. Gonfalone, J. Phys. (Paris) 33, 521 (1972).
[7]L. R. O. Storey and J. Thiel, Phys. Fluids 21, 2325 (1978).
[8]T. H. Stix, *The Theory of Plasma Waves* (McGraw–Hill, New York, 1962), Chap. 9, p. 225.
[9]H. H. Kuehl, Phys. Fluids 17, 1275 (1974).
[10]R. Pottelette, Ann. Geophys. 29, 201 (1973).

Expansion of electron bunches

N. R. Pereira and R. O. Hunter, Jr.

Western Research Corporation, San Diego, California 92121
(Received 31 October 1980; accepted 29 July 1981)

The expansion of an electron bunch is solved exactly in Lagrangian coordinates, and the result is compared to a self-similar solution.

One of the simplest plasma physics phenomena is the one-dimensional expansion due to self-electric fields of an initially localized electron bunch. This situation is not only of theoretical interest; for example, deeply modulated electron beams can be considered as a train of these bunches. The modulation depth of the beam deteriorates when the bunches have widened beyond the original inter-bunch distance, and this process can be a limit to the beam current.

The electron bunches are assumed to be created at rest in a magnetic field strong enough to make the dynamics one-dimensional. The nonlinear fluid equations for the electrons (ions are absent, there is no charge neutralization) can be treated in various ways. One way is to use Lagrangian coordinates; this approach has been applied to large-amplitude electron plasma oscillations,[1-4] but apparently not yet to non-neutral charge bunches. Another way is to attempt a self-similar solution.[5] The two calculations give a simple and interesting example of the utility and limits of self-similar solutions.[6]

The equations for a one-dimensional single species cold electron fluid are the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0\,, \tag{1a}$$

the momentum equation

$$\frac{\partial p}{\partial t} + v\,\frac{\partial p}{\partial x} = -e\,\frac{\partial \phi}{\partial x}\,, \tag{1b}$$

and Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} = -\,\frac{en}{\epsilon_0}\,. \tag{1c}$$

Here, $n(x, t)$ and $p(x, t)$ are the electron density and momentum, e is the electron charge, ϕ is the electric potential, and ϵ_0 is the permittivity of vacuum. In addition, when considering a charge bunch the charge/area is constant,

$$e\int n(x, t)dx = \text{const}\,. \tag{2}$$

These equations are linearized in Lagrangian coordinates τ and ξ; $\tau = t$, and

$$x = \xi + \int_0^\tau d\tau'\ v(\xi, \tau')\,. \tag{3}$$

Initially, ξ is the same as x, but as the system evolves the coordinate ξ remains fixed in the fluid, and the coordinate x must be found by backtracking the fluid motion. The convective derivative following the fluid is

$$\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \tau}\,, \tag{4}$$

and the space element becomes simply $d\xi = (\partial x/\partial \xi)^{-1}dx$. The continuity equation expresses conservation of mass between two neighboring positions ξ and $\xi + d\xi$; therefore,

$$n(\xi, \tau) = n_0(\xi)\left(1 + \int_0^\tau d\tau'\ \frac{\partial v(\xi, \tau')}{\partial \xi}\,d\tau'\right)^{-1}, \tag{5}$$

 0031-9171/81/112122-02$00.90

where $n_0(\xi)$ is the initial electron density. The momentum equation (1b) becomes

$$\frac{\partial p(\xi,\tau)}{\partial \tau}=eE(\xi,\tau), \qquad (6)$$

where the electric field $E(=-\partial\phi/\partial x)$ is localized in the fluid,

$$\frac{\partial E(\xi,\tau)}{\partial \tau}=0. \qquad (7a)$$

To derive this equation algebraically, one can use Eq. (1c) supplemented by the time derivative of (1c), (1a), and the spatial integration; alternatively, one can realize that the field at ξ is given by the total charge to the left of ξ, which is invariant. In terms of the initial charge density, the field is given by Poisson's equation

$$\frac{\partial E(\xi)}{\partial \xi}=\frac{en_0(\xi)}{\epsilon_0}. \qquad (7b)$$

The dynamical equations are now linear, and can be solved exactly once the initial conditions are given.

First, consider a charge bunch initially at rest without an external field. The appropriate initial condition is

$$n_0(\xi)=NF(\xi/s)/s, \qquad (8a)$$

where N is the number of particles per unit area, s measures the width of the bunch, and $F=F(y)$ gives the functional form of the bunch with integral normalized to unity. An example is $F=\pi^{-1/2}\exp(-y^2)$. The initial electric field E_0 vanishes in the center of the bunch, and is, therefore, given by

$$E_0(\xi)=\frac{Ne}{\epsilon_0}\,g\left(\frac{\xi}{s}\right), \qquad (8b)$$

where

$$g(y)=\int_0^y F(y)dy.$$

$E_0(\xi)$ has width s, and increases monotonically with ξ from $-Ne/2\epsilon_0$ to $+Ne/2\epsilon_0$.

The momentum now follows easily from Eq. (6) and (8):

$$p(\xi,\tau)=\frac{e^2N\tau}{\epsilon_0}\,g\left(\frac{\xi}{s}\right). \qquad (9a)$$

The position in Eulerian coordinate then becomes for nonrelativistic velocities $v=p/m$ (m is the electron mass)

$$x=\xi+\frac{e^2N\tau^2}{2m\epsilon_0}\,g\left(\frac{\xi}{s}\right). \qquad (9b)$$

The exact formula for the density, Eq. (5), can now be written down. The density is an explicit function of ξ, but an explicit result in terms of x for arbitrary initial shapes must be found numerically from the generally transcendental relation (9b).

The self-similar solution[5] for an expanding charge bunch uses as self-similar variables $\zeta=x/t^2$, $N(\zeta)=nt^2$, $V(\zeta)=v/t$, and $\Phi(\zeta)=\phi/t^2$. Substitution in Eq. (1) gives

$$-2N-2\zeta\frac{dN}{d\zeta}+\frac{d(NV)}{d\zeta}=0, \qquad (10a)$$

$$V-2\zeta\frac{dV}{d\zeta}+V\frac{dV}{d\zeta}+\frac{e}{m}\frac{d\Phi}{d\zeta}=0, \qquad (10b)$$

$$\frac{d^2\Phi}{d\zeta^2}=-\frac{e}{\epsilon_0}N. \qquad (10c)$$

Equation (10a) can be integrated once, to give

$$(V-2\zeta)N=\text{const}. \qquad (11)$$

The bunch is initially symmetric; therefore, $V(\zeta)=V(-\zeta)$, or $V(\zeta=0)=0$, and the constant in Eq. (11) equals zero. Then,

$$V=2\zeta. \qquad (12)$$

Substituting this in Eq. (10b) and integrating gives

$$(e/m)\Phi=\text{const}-\zeta^2, \qquad (13a)$$

and into Eq. (10c),

$$N=2\epsilon_0 m/e^2. \qquad (13b)$$

This constant density is obviously an unacceptable solution for an expanding bunch, because the change density is not localized, and the total charge is infinite.

It is clear from Eq. (9b) why the self-similar assumption gives an unacceptable solution. The self-similar variable is $\zeta=x/t^2$, but (9b) is

$$\zeta=\frac{\xi}{t^2}+\frac{e^2}{2\epsilon_0 m}\,g\left(\frac{\xi}{s}\right); \qquad (14)$$

the time t cannot be removed from this expression, except for $\xi=0$, or in the limit of large times. Indeed, for large times the expanding electron bunch has a large scale length, and the charge density is almost constant, in agreement with Eq. (13b) from the self-similar analysis. Equation (14) is an exceedingly simple example of an asymptotically self-similar transformation.[6]

In conclusion, it should be mentioned that the Lagrangian analysis can be extended to two and three dimensions.[7] Also temperature, relativistic dynamics, and external electric fields can be included.

[1]J. Dawson, Phys. Rev. **113**, 383 (1959).

[2]P. A. Sturrock, Proc. R. Soc. London Ser. A **242**, 177 (1957).

[3]E. A. Jackson, Phys. Fluids **3**, 831 (1960).

[4]R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York 1974), Chap. II and references therein.

[5]H. Shen and K. E. Longren, IEEE Trans. Plasma Sci. **PS-4**, 144 (1976).

[6]G. I. Barenblatt, *Similarity, Self-Similarity and Intermediate Asymptotics* (Consultants Bureau, New York, 1979), especially Chap. V.

[7]D. D. Holm, S. F. Johnson, and K. E. Longren, Bull. Am. Phys. Soc. **25**, 843 (1980), and Appl. Phys. Lett. **38**, 519 (1981).