Quality and Quantity, 12 (1978) 175–178

A RELATION BETWEEN THE GINI AND ELTETO MEASURES OF INEQUALITY

NINO R. PEREIRA

Lawrence Berkeley Laboratory, University of California, Berkeley, California, 94720

PATRICIA WILSON SALINAS

City of San Francisco Planning Department, 100 Larkin Street, San Francisco California, 94102

Single measures for characterising the overall inequality of distributions have generally been derived from cumulative Lorenz curves. The most popular of these measures is the Gini index of inequality (Alker and Russet, 1964). The Gini index requires knowledge of the complete Lorenz curve, and consequently necessitates rather detailed data.

Elteto and Frigyes (1968) proposed other measures of inequality of distributions, which are easier to compute and are more compatible with grouped data. Nevertheless, the Elteto measures are seldom used. The aim of this paper is to describe, mainly in geometrical terms, various relationships between the Elteto measures and the Gini index of inequality. It will turn out that the relevant quantity is the relative mean deviation, known as "maximum equalisation percentage" T, which has been proposed as an inequality measure many times in the past (Kondor, 1971). However, to our knowledge the geometrical connections between the various measures have not been given. For definiteness of terminology, we take an income distribution as our example.

The Elteto measures u, v, and w are defined as

$$u = m/m_1, \qquad v = m_2/m_1, \qquad w = m_2/m \tag{1}$$

where m is the mean income of a given population, m_1 is the mean income of those with an income smaller than m, and m_2 is the mean income of those with incomes larger than m. Only two of these measures are independent, because $uw = v$. The range of u, v, and w is from one to infinity, and therefore we define the standardised meaures u', v', and w',

$$\begin{aligned} u' &= 1 - 1/u = (m - m_1)/m, \\ w' &= 1 - 1/w = (m_2 - m)/m_2, \\ v' &= 1 - 1/v = (m_2 - m_1)/m_2. \end{aligned} \tag{2}$$

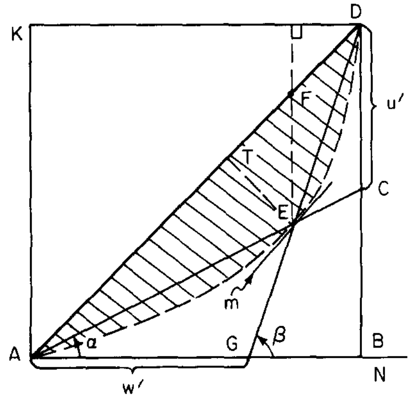

Fig. 1. The Lorenz curve with the auxiliary lines defined in the text.

We also have, using $v = uw$,

$$v' = 1 - (1 - u')(1 - w'). \qquad (3)$$

The value of the primed measures lies between 0 and 1.

The graphical meaning of these measures is given in Fig. 1. On the given Lorenz curve (shown by the dotted line) the "equal share point" (E), where the income is m, is found at the point where the tangent to the curve has a slope of one, i.e. the normalised mean income. If all income were shared equally, the Lorenz curve would degenerate to the diagonal AD with slope one.

The slope of the solid line AE, the average income of those earning below the mean, is given by the angle α, while the slope of the solid line ED, the average income of those earning above the mean, is measured by β. Thus the Elteto measures can be expressed as $u = 1/\tan(\alpha)$, $w = \tan(\beta)$, and v as the product of these. The normalised measures u' and w' are then $u' = 1 - \tan(\alpha)$ and $w' = 1 - \cot(\beta)$.

From Fig. 1 we see that u' is the length of the line from point D to point C, because the line from B to C is equal to $\tan(\alpha)$ times the length AB, which is unity. Likewise, the index w' is the line AG.

Obtaining a geometrical representation of v' is less direct. We see from triangle ACB in Fig. 2 that $EH = AH(1 - u')$, and from triangle GBD that $GH = EH(1 - w') = AH(1 - u')(1 - w')$. Thus, with $w' = AH - HG$ and eqn. (3) it follows that

$$w' = AH \times v'. \qquad (4)$$

The surface of rectangle $AGTK$ is another representation of w'. Thus, v' can be represented by the length of the line AL. The rectangle formed

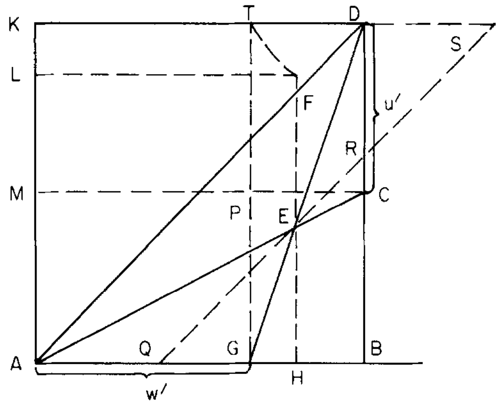

Fig. 2. The geometrical significance of the Elteto measures as discussed in the text.

by AL and the line AH has the same area as rectangle $AGTK$. Alternatively, u' is the surface of rectangle $MCDK$, analogous to w' above. Then $1 - v' = (1 - w')(1 - u')$ is given by the rectangle $GBCP$, and v' by the L-shaped figure $AGPCDK$.

How do we relate the Gini measure of inequality to u', w', and v'? The Gini coefficient of inequality G is defined by normalising A_G, the "area of inequality," i.e. the area between the Lorenz curve and the diagonal line of equality (shaded in Fig. 1). Thus the Gini index G is twice the area of inequality and lies between 0 and 1. Using the Elteto parameters we can choose a rather rough but convenient approximation, A_e, to the area of inequality A_G, namely the surface of triangle AED in Fig. 1. As the area A_e equals one half times the height EF (times the base, unity), the approximated Gini coefficient based on this estimate, G_e, is

$$G_e = EF. \tag{5}$$

From triangle ACD in Fig. 2 we see that $EF = u' \times AH$, or, using eqn. (4),

$$G_e = u'w'/v'. \tag{6}$$

Thus G_e is equal to the "maximum equalisation percentage", defined in Kondor (1971) as $T = 1/2m \times [\Sigma_{i=1}^{n} |x_i - m|]/n$, where (x_i) is the income vector, $i = 1, 2, ..., n$.

Obviously, G_e is smaller than G, as triangle AED lies within the inequality area A_G. However, we can give an upper estimate for G, G_E, by observing in Fig. 2 that A_G always lies inside the trapezoid $AQRD$. Thus G_E is twice the surface of this trapezoid. The height h of the trapezoid is the same as that of triangle *AED* (*TE* in Fig. 1). Its surface is that of parallelogram *AQSD*, twice that of triangle *AED*, minus the surface of triangle *RSD*. This surface equals half of the product of *DR* and *DS* or $h\sqrt{2}$. Thus, $G_E = 2G_e - G_e^2$, and

$$G_e \leqslant G \leqslant 2G_e - G_e^2. \tag{7}$$

The general proportionality between G and G_e is evident from this equation.

The purpose of the Gini coefficient, or other measures such as G_e, is to describe the inequality in an income distribution in a single number. One number, however, can only give a limited knowledge of the original distribution. The geometrical considerations above suggest to us a more accurate description of an income distribution: Namely, we can perform the same series of steps that gave us the Elteto coefficients in the first place, on the part of the Lorenz curve to the left and right of the equal share point *E*. The equal share point of the left subsection gives us an "equal subshare point," where the derivative of the Lorenz curve equals the slope of the line *AE*, the average income of those earning below the mean; the right subsection has an "equal share point" where the derivative of the Lorenz curve is equal to the slope of the line *ED*. If this additional information is given, we can, among other things, make a much better estimate of the Gini coefficient G.

In conclusion, we have shown some geometrical relations between the Elteto measures, and between these and the Gini index of inequality G. An estimated Gini coefficient G_e is shown to be in general proportional to G. As a final remark, let us note that a wealth of different representations in terms of various triangles, angles, etc. can be given both for the Elteto measures and for the Gini coefficient, besides the ones above.

References

Alker, H.R., Jr. and Russet, B.M. (1964) "On Measuring Inequality," *Behavioral Science* 9, 207–218.

Elteto, O. and Frigyes, E. (1968). "New Income Inequality Measures as Efficient Tools for Causal Analysis and Planning," *Econometrica* 36: 383–396.

Kondor, Y. (1971). "An Old-New Measure of Income Inequality," *Econometrica* 39: 1041–1042. Kondor summarizes the history of the discovery and rediscovery of *T*, from its Italian origins around the turn of this century to Kuznets and Schutz, who are among the economists to have rediscovered the measure in the 1950's.