Nonlinear Schrödinger equation including growth and damping

N.R. Pereira and Lennart Stenflo[a)]

Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 29 March 1977)

The nonlinear Schrödinger equation, with complex coefficients that describe growth and damping, is considered. An exact stationary soliton solution is found for arbitrary growth and damping strength.

The nonlinear Schrödinger equation, in normalized form

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi=0\ , \tag{1}$$

approximates the nonlinear evolution of large-amplitude dispersive and weakly nonlinear systems.[1-3] Examples of current interest in plasma physics are Langmuir turbulence[4] and nonlinear propagation of lower hybrid waves in tokamaks.[5] In these examples, ψ stands for the complex envelope of the high-frequency electric field $E(x,t)=\mathrm{Re}\psi(x,t)\exp(-i\omega t)$, with ω being the reference frequency. Then, the integral $W\equiv\int|\psi|^2dx$ represents the field energy, which is conserved by Eq. (1).

In this paper we want to study Eq. (1) with the addition of simple terms that take growth and damping into account. The growth could be responsible for the presence of waves that are large enough to necessitate the nonlinear term, whereas the damping might make an ultimate stationary state possible.

Accordingly, we modify Eq. (1) by adding a linear growth rate $\gamma_k=\gamma_0-\gamma_2k^2$ for the Fourier mode of ψ with wavenumber k. (We choose the constants to be positive.) This model of growth rate typically[6] gives rise to unstable solutions in the mode coupling equations,[7] which are superficially similar to Eq. (1). By Fourier transforming Eq. (1) and neglecting the nonlinear term, it is clear that this modification adds a term $+i\gamma_0\psi+i\gamma_2\partial^2\psi/\partial x^2$ to the right side of Eq. (1).

In addition, we allow for the possibility of a nonlinear (amplitude-dependent) damping, by including a term $-i\gamma_n|\psi|^2\psi$. This damping might be caused by collisions in the absence of lower order effects,[8] or by trapped particles.[9] With the various growth and damping terms our model becomes Eq. (1) with complex coefficients

$$i\left(\frac{\partial}{\partial t}-\gamma_0\right)\psi+(1-i\gamma_2)\frac{\partial^2\psi}{\partial x^2}+(1+i\gamma_n)|\psi|^2\psi=0\ . \tag{2}$$

The first integral W is not conserved by Eq. (2); instead, we have

$$\frac{1}{2}\frac{d}{dt}W=\gamma_0W-\gamma_2\int\left|\frac{\partial\psi}{\partial x}\right|^2dx-\gamma_n\int|\psi|^4dx\ . \tag{3}$$

For the other integral invariants[3] of Eq. (1), similar relations can be found.[10,11]

We treat two aspects of Eq. (2), viz., (i) the approach to equilibrium[7] of an assumed solution, and (ii) an exact solution in equilibrium. We start with the time-dependent problem. A good[10] approximation to a particular solution for small γ_0, γ_2, and γ_n is the solitary standing wave

$$\psi(x,t)=\sqrt{2}\,K(t)\exp\left[i\int_0^t K^2(t')\,dt'\right]\mathrm{sech}[K(t)x]\ , \tag{4}$$

where the single parameter $K(t)$ represents both amplitude and width, and determines the nonlinear frequency shift. Substitution of (4) in (3) leads to

$$\tfrac{1}{2}(dK/dt)=\gamma_0K-(\gamma_2+4\gamma_n)K^3/3\ . \tag{5}$$

It is clear that an equilibrium ($dK/dt=0$) between growth and damping is reached when

$$K^2=K_e^2\equiv 3\gamma_0/(\gamma_2+4\gamma_n)\ . \tag{6}$$

For small $K\ll K_e$, the growth of K with time is exponential, but K decreases as $t^{-1/2}$ for $K\gg K_e$. The full time dependence of K is readily found as

$$K^2(t)=\frac{K^2(0)\exp(4\gamma_0t)}{1+K^2(0)(\gamma_2+4\gamma_n)[\exp(4\gamma_0t)-1]/3\gamma_0} \tag{7}$$

(cf. Ref. 7, Fig. 8, for a plot of a similar function).

We note that $K(t)$ can grow explosively, $K^2\sim(t-t_0)^{-1}$, if $\gamma_2+4\gamma_n<0$, i.e., with dispersive or nonlinear growth rather than damping. The explosion occurs when the denominator in (7) vanishes at $t=(4\gamma_0)^{-1}\ln[1-K_e^2/K^2(0)]$. Ultimately higher order effects, such as terms like $|\psi|^4\psi$ or $\partial^4\psi/\partial x^4$ which are neglected in our model, will saturate such an instability.

We now present an exact solution of Eq. (2) in equilibrium. As can be verified by direct substitution,

$$\psi(x,t)=\sqrt{2}\,L\,[\mathrm{sech}(Kx)]^{1+i\alpha}\,e^{-i\Omega t} \tag{8}$$

satisfies Eq. (2) exactly, for arbitrary γ_0, γ_2, and γ_n, by an appropriate choice of the equilibrium amplitude L, width K^{-1}, and nonlinear frequency shift Ω. The imaginary part of the complex exponent is found as

$$\alpha=-\beta+(2+\beta^2)^{1/2}\ ,$$
$$\beta=\tfrac{3}{2}[(1-\gamma_2\gamma_n)/(\gamma_2+\gamma_n)]\ . \tag{9a}$$

To first order in the damping and growth constants γ we find

$$\alpha\approx\tfrac{2}{3}(\gamma_2+\gamma_n)\simeq 1/\beta\ . \tag{9b}$$

The inverse width is given by

[a)]Permanent address: Umea University, 90187 Umea, Sweden.

$$K^2 = \frac{\gamma_0}{2\alpha - \gamma_2 + \alpha^2\gamma_2} \tag{10a}$$

$$= K_e^2\left(1 + \frac{\alpha}{3}\cdot\frac{2\alpha\gamma_n - \alpha\gamma_2 - 6\gamma_2\gamma_n}{2\alpha - \gamma_2 + \alpha^2\gamma_2}\right), \tag{10b}$$

where K_e^2 is the equilibrium value of Eq. (6).

The amplitude L is

$$L^2 = K^2(1 + \tfrac{3}{2}\alpha\gamma_2 - \tfrac{1}{2}\alpha^2), \tag{11}$$

while the nonlinear frequency shift turns out to be

$$\Omega = -K^2(1 + 2\alpha\gamma_2 - \alpha^2). \tag{12}$$

How do growth and damping affect this equilibrium soliton as compared to the stationary solution without growth and damping? Equations (8)–(12) reduce to the well-known soliton $\psi_0(x,t) = \sqrt{2}\,K\,\mathrm{sech}(Kx)\exp(iK^2t)$ when the growth constant γ_0 as well as the damping constants γ_2 and γ_n vanish. The complex exponent α appears to first order in γ. Thus, ψ_0 is modified by the appearance of an x-dependent phase of $[\mathrm{sech}Kx]^{i\alpha} \equiv e^{i\phi(x)}$, i.e., $\phi(x) = \alpha\ln\mathrm{sech}(Kx)$.

The physical meaning of this phase follows from the equation for the energy density $|\psi|^2$ obtained from Eq. (2)

$$\frac{\partial|\psi|^2}{\partial t} + \frac{\partial}{\partial x}\left\{|\psi|^2\left[\frac{\partial}{\partial x}(2\phi - \gamma_2\ln|\psi|^2)\right]\right\} = 2\gamma_0|\psi|^2 - \gamma_n|\psi|^4 - \gamma_2\left|\frac{\partial\psi}{\partial x}\right|^2. \tag{13}$$

The first term on the right-hand side of Eq. (13) is positive, and represents an energy source. The last terms are negative, and describe energy sinks. In equilibrium, there is no net energy production, i.e., $\partial\int|\psi|^2dx/\partial t = 0$. However, as the energy source strength differs from the sink strengths, as functions of x, energy must flow toward the region of stronger absorption of energy. This flow is given by the term in braces on the left-hand side of (13). For small damping, the expression in brackets, the flow velocity v_g, reduces to

$$v_g = \tfrac{2}{3}(\gamma_2 - 2\gamma_n)K\tanh Kx. \tag{14}$$

For nonlinear damping only, $\gamma_n > 0$, $\gamma_2 = 0$, the flow is toward the origin, while for dispersive damping, $\gamma_2 > 0$, $\gamma_n = 0$, the flow is in the opposite direction.

To second order in γ, the growth and damping change the nonlinear frequency shift Ω, and affect the usual relation between amplitude L and width K^{-1}. Without damping, $L = K$, but to second order in γ we have from (11)

$$L^2 = K^2[1 + (\gamma_2 + \gamma_n)(7\gamma_2 - 2\gamma_n)/9]. \tag{15}$$

For nonlinear damping only the amplitude L is lower than for the undamped soliton with the same width. The damping goes like $|\psi|^4$, is large where $|\psi|$ is large, and tends to decrease the amplitude. For dispersive damping only, the soliton is more peaked than an undamped one, because now the damping occurs mainly in the soliton sides where $|\partial\psi/\partial x|^2$ is substantial.

Our previous estimate (6) for the equilibrium amplitude K_e is correct to first order, as is clear from Eq. (10b), but can be larger or smaller than the actual value K to second order.

Finally, we note that our standing equilibrium soliton (8)–(12) could easily be generalized to a moving one by a Galilean transformation $x' = x - vt$, to an inhomogeneous medium by the transformation mentioned in Ref. 12, and that also an exact periodic solution can be found.

In conclusion, we have introduced the nonlinear Schrödinger equation with complex coefficients by considering model growth and damping processes. We have first, by approximate methods, treated the approach to equilibrium of a localized solution (soliton). Subsequently, we have found an exact equilibrium soliton. This soliton differs from the usual undamped soliton in a phase proportional to growth and damping parameters γ. To higher order in γ the amplitude and width are also affected, but the characteristic sech shape is not changed.

After completion of this work we noted that Eq. (2), with nonlinear growth only ($\gamma_2 = 0$, $\gamma_n < 0$) and with different initial conditions, had already been solved exactly.[13] This solution is rather complicated, but can be simplified by introducing a complex exponent as done here (and in Ref. 14).

ACKNOWLEDGMENTS

We are grateful to A. N. Kaufman for encouragement and useful discussions. One of us (L. S.) also thanks him for his gracious hospitality.

This work was done under the auspices of the U. S. Energy Research and Development Administration.

[1]G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974), Chap. 16.

[2]V. I. Karpman, *Nonlinear Waves in Dispersive Media* (Pergamon, London, 1973), Sec. 27.

[3]A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE **61**, 1443 (1973).

[4]N. R. Pereira, J. Denavit, and R. N. Sudan, Phys. Fluids **20**, 271 (1977).

[5]H. H. Kuehl, Phys. Fluids **19**, 1972 (1976), G. Johnston (private communication).

[6]W. M. Manheimer, Naval Research Laboratory, Memo No. 3369 (1976).

[7]E. Ott, W. M. Manheimer, D. L. Book, and J. P. Boris, Phys. Fluids **16**, 855 (1973).

[8]L. Stenflo, Z. Phys. **239**, 184 (1970), P. K. Kaw, C. Z. Cheng, and L. Chen (to be published.)

[9]J. Canosa, Phys. Fluids **19**, 1952 (1976); see also B. Buti, IEEE Trans. Plasma Sci. **PS-4**, 292 (1976).

[10]N. R. Pereira, Phys. Fluids **20**, 1735 (1977).

[11]D. R. Nicholson and M. V. Goldman, Phys. Fluids **19**, 1621 (1976), as well as references therein.

[12]H. H. Chen and C. S. Liu, Phys. Rev. Lett. **37**, 693 (1976).

[13]R. Haberman, J. Fluid Mech. **58**, 129 (1973).

[14]R. Haberman, SIAM J. Appl. Math. **32**, 154 (1977).