Collisions between Langmuir solitons

N. R. Pereira*

Laboratory of Plasma Studies, Cornell University, Ithaca, New York 14853
(Received 6 July 1976)

Collisions between two equal Langmuir solitons with opposite velocities are studied numerically, as a function of soliton strength, velocity, and relative phase angle. The computations show that all three parameters are important in the outcome of a collision, for which qualitative estimates are given.

I. INTRODUCTION

Strong electrostatic turbulence has been described[1,2] in terms of "Langmuir" solitons. These solitons are nonlinear solitary wave solutions of the normalized one-dimensional equations[3,4]

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}-NE=0, \tag{1}$$

$$\frac{\partial^2 N}{\partial t^2}-\frac{\partial^2 N}{\partial x^2}=\frac{\partial}{\partial x^2}|E|^2. \tag{2}$$

Here, we use the notation and normalizations of Ref. 4: E is the (envelope of the) electrostatic field, and N is the relative perturbation of the background ion density.

The stationary wave (single soliton) solution of Eqs. (1) and (2) is

$$E_1=K[2(1-v^2)]^{1/2}[\cosh K(x-p-vt)]^{-1} \times\exp[iv(x-p)/2-i\Omega t], \tag{3}$$

$$N_1=-2K^2/\cosh^2 K(x-p-vt), \tag{4}$$

where $\Omega=v^2/4-K^2$, K is a strength parameter, v is the group velocity, and p is the initial position. Setting $N=-|E|^2$, i.e., neglecting the time derivative of N in Eq. (2), yields the well-known nonlinear Schrödinger equation

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}+|E|^2E=0, \tag{5}$$

which has the soliton solution (3) with the factor $1-v^2$ set to unity. This equation is exactly solvable with the inverse scattering method,[5] in which solitons are basic elements.[6] These solitons keep their identity in nonlinear collisions with one another: hence, the name "soliton."[7] The soliton theory of Langmuir turbulence also uses the stability of solitons on collision,[1] or assumes an estimated merging rate.[2]

In this paper we study the collision between two equal solitons numerically.[2,8] Section II gives a brief mathematical introduction. In Sec. III we present the computational results, which show that solitons can pass through one another, merge to form one large soliton, or can exhibit some intermediate behavior. We find that the relative phase difference between the electric fields is also important. Estimates for the parameter dependence of the various outcomes of a collision are given in Sec. IV.

In a previous paper,[4] we showed that particle simulations of Langmuir solitons compare favorably to the model equations (1) and (2), provided that an approximate Landau damping is included, but not without the damping term. It is still of interest to investigate Eqs. (1) and (2), however, both as an approximation to the damped equations and for mathematical reasons.

II. MATHEMATICAL PROPERTIES

For future reference in this section we present the mathematical properties of Eqs. (1) and (2).

We see that Eqs. (1) and (2) are invariant under space reversal $x\to -x$, and under time reversal with complex conjugation. Thus, solutions have the same invariance, provided the initial condition is also invariant. We do not have invariance under a scale transformation[9]

$$x'=ax,\quad t'=a^h t, \qquad E=a^m E'(x',t'),\quad N=a^n N'(x',t'). \tag{6}$$

We can choose $2m=n$, $n=2$, which leaves Eqs. (1) and (2), respectively invariant, but no unique choice for h is possible. A convenient choice is $h=1$, which leaves velocities invariant. With the soliton parameter K as scaling factor, Eq. (1) becomes, suppressing the primes,

$$\frac{1}{K}i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial^2 x}-NE=0. \tag{7}$$

Instead of studying Eqs. (1)–(4) we can study a unit soliton, Eqs. (3) and (4) with $K=1$, in Eqs. (2) and (7). Only the phase of the unit soliton, $v'x'/2-\Omega't'$, with $v'=v/K$ and $\Omega'=(v'^2/4-1)K$, corresponding to the time derivative in (7), are functions of K.

When $K\gg 1$, the time derivative on E is large, as the terms ∂^2E/∂^2x and $-NE$ are of order unity. However, the time derivative of N in Eq. (2) is always about 1, so that changes in the potential $N(x,t)$ in Eq. (7) are slow compared with changes in E. When K is of order one, the time derivatives are of the same order, and interference between oscillations described by the two equations is possible.

The wave equation (2) leaves the total ion density perturbation $\int N\,dx$ invariant. Consequently, the invariant $\int \partial N/\partial t\,dx$ is zero. The Schrödinger equation (1) has a mass invariant, the total electrostatic energy

$$W=\int |E|^2\,dx, \tag{8}$$

and a momentum invariant, which is zero for our initial condition. When combined with (2) an energy invariant

TABLE I. Invariants for the single soliton of Eqs. (3) and (4). The interaction energy S_I is calculated for ion density and electrostatic energy shifted over a distance d $(a = Kd)$.

(1) Mass invariant: $W = \int dx \mid E \mid^2 = 4K(1-v^2)$.

(2) Momentum invariant $P = \int dx\, i(EE_x^* - E^*E_x) = 4Kv(1-v^2)$.

(3) (a) Total energy ion wave: $S_N = \int dx\,(N^2/2 + V^2/2)$

$$= 8/3K^3(1+v^2).$$

(b) Kinetic energy ion wave: $S_K = \int dx\, V^2/2 = 8/3\,K^3 v^2$.

(4) Field kinetic energy $S_E = \int dx \mid \partial E/\partial x \mid^2$

$$= K/3\,(1-v^2)\,(4K^2 + 3v^2).$$

(5) Energy invariant $S = 4K^3/3(5v^2 - 1) + v^2(1-v^2)K$,

$$= [(4K^2/3)\,(4K - 5W/4) + W(1-W)/(4K)]/4.$$

(6) Interaction energy $S_I = \int dx N(z) \mid E \mid^2 (z + a)$

$$= -16K^3(1-v^2)\left(\frac{a\cosh a}{\sinh a} - 1\right)\Big/\sinh^2 a$$

$$\simeq -16/3K^3(1-v^2)\,(1 - 2/5\,a^2).$$

exists

$$S = \int (N|E|^2 + |\partial E/\partial x|^2 + N/2 + V^2/2)dx, \tag{9}$$

with $\partial N/\partial t = -\partial V/\partial x$. The terms in S represent interaction energy, kinetic energy of E, and ion energy. They are calculated for a single soliton in Table I. More invariants do not seem to exist, in contrast to Eq. (5), which has an infinite series of invariants.

The occurrence of a linear interaction, when Eqs. (1) and (2) couple weakly, or a nonlinear interaction, when the coupling between (1) and (2) is strong, could be determined by comparing the interaction energy of a single soliton to, for example, its ion energy. The estimates for the separation between the different outcomes of a collision based on these comparisons (Sec IV) are in reasonable agreement with the observed separations.

III. NUMERICAL COMPUTATIONS

The numerical computations presented in this section were performed with a code using a Fourier transform method on a periodic system of length 16 K^{-1}. To start the computations we must specify, in addition to $E(t=0)$ and $N(t=0)$, the time derivative $M = \partial N/\partial t$ at $t=0$, because Eq. (2) is second order in time. For a soliton (4) with velocity v, $M_1 = v\,\partial N_1/\partial x$. The initial condition for our two solitons is then

$$E_2 = E_1(v, p;\ t=0) + E_1(-v, -p;\ t=0)e^{i\phi}, \tag{10a}$$

$$N_2 = N_1(v, p;\ t=0) + N_1(-v, -p;\ t=0), \tag{10b}$$

$$M_2 = M_1(v, p;\ t=0) + M_1(-v, -p;\ t=0), \tag{10c}$$

where the index 2 indicates the total field, ion density, etc., of two solitons. With this initial condition N_2 and M_2 are symmetric around $x=0$. The electric field E_2 is only symmetric when $\phi = 0$, and is antisymmetric when $\phi = \pi$. The initial condition has three free parameters, the strength parameter K, the group velocity v, and the phase angle between the electric fields of the two solitons ϕ. The initial position p does not matter, as long as the initial solitons are well separated.

Depending on the parameters, the solitons could: (i) reflect from or pass through each other, (ii) lose their identity and spread out through the system, or (iii) merge to form one larger and narrower soliton with additional ion sound waves. The different possibilities are shown in Figs. 1–6 for one strength parameter $K = 1.5$. The initial group velocity v and the phase angle ϕ are given in each figure.

Possibility (i) is illustrated in Fig. 1. The initial condition (10) Fig. 1(a), appears the same for all collisions. The electrostatic energy $|E|^2$ is independent of ϕ, for well-separated solitons, and is proportional to $1 - v^2$. Thus, the velocity only shows up in the scale of $|E|^2$.

At $t = 2.5$, Fig. 1(b), one narrow, high structure has been formed, seemingly very similar to a single stationary soliton. However, the maxima of $|E|^2$ and N have a different ratio than in a stationary soliton with $v = 0$. Note that the maxima of the field and the ion density are larger than twice the initial value. The width of $|E|^2$ is narrower than in a stationary soliton, which follows from conservation of total energy W.

At $t = 3.5$ Fig. 2(c) the beginning of the separation is seen. The field energy is still one structure, but the

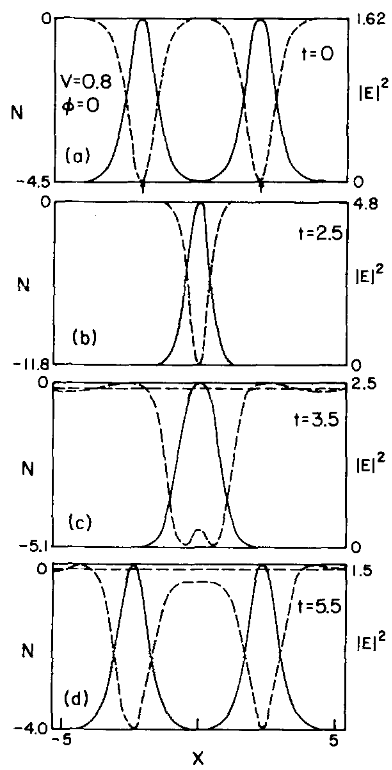

FIG. 1. The electrostatic energy density $|E|^2$ (solid line), and the ion density perturbation N (broken line) for a collision between two equal solitons with opposite velocities. The scale on the right is for $|E|^2$, on the left for N. The strength parameter $K = 1.5$, the velocity $v = 0.8$, and the phase difference $\varphi = 0$.

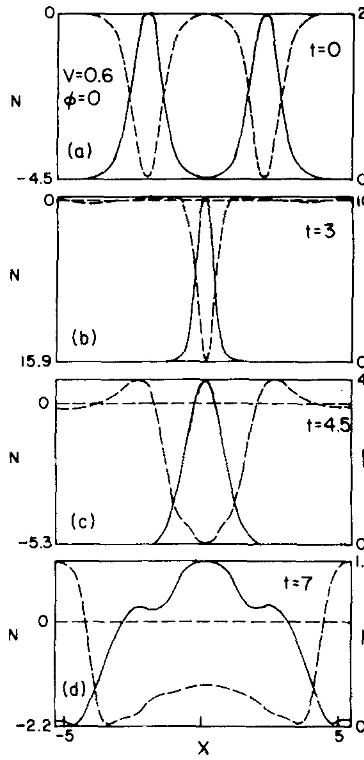

FIG. 2. The electrostatic energy density $|E|^2$ (solid line), and the ion density perturbation N (broken line) for a collision between two equal solitons with opposite velocities. The scale on the right is for $|E|^2$, on the left for N. The strength parameter $K=1.5$, the velocity $v=0.6$, and the phase difference $\varphi=0$. Note that the scale is different from the previous figure.

ion density is splitting in two in the middle. The solitons emerging in the final state (d) are wider but smaller in amplitude than the original ones, and some ion sound has appeared.

The second possibility is shown for the same K but smaller velocity, $v=0.6$, in Fig. 2. Situations similar to those of Figs. 1(b), (c), and (d) are shown. The solitons now move more slowly than in the previous case, so that corresponding moments in the collision occur later. Figure 2(b) at $t=3$ is very much like Fig. 1(b), apart from the values of the maxima, which vary rapidly during the collision. At $t=4.5$, the field and ion density are wider and smaller than in Fig. 1, and no sign of splitting is seen. This broad structure persists for subsequent times, shown in Fig. 2(d) at $t=7$. Sound waves are also present here.

Merging solitons are shown in Fig. 3 for smaller velocity, $v=0.4$. The structure at $t=3.5$ is very high and narrow. The splitting of the ion density occurs at $t=5.5$ in Fig. 3(c). Peaks appear in the ion density at the sides of $|E|^2$. The peaks travel away from the resulting stable soliton. This soliton is a little over twice as high as the initial ones, and maximum and width exercise damped oscillations around average values.

The phase difference ϕ has a marked influence on the interaction. Figure 4 shows the merging solitons of Fig. 3, but with phase difference $\phi=\pi$, i.e., antisymmetric electric fields. The electric fields of the two solitons now repel each other as seen in (b) and (c), while the ion densities overlap. After a long time during which the situation is almost stationary the solitons separate again, as seen in (d).

For a group velocity $v=0.7$, the ion densities still show separation; however, when the group velocity is increased to $v=0.9$ the ion densities come together completely, as observed in Fig. 5(a). An intermediate case, $\phi=\pi/2$, with the same parameter K and $v=0.4$ is shown in Fig. 6. Now, one large and one small soliton emerge and ion waves appear as always.

Whether solitons pass through one another or merge on collision is summarized in Fig. 7, where the different possibilities for phase angle $\phi=0$ are plotted in K-v parameter space. Parameters for which solitons pass through each other, as in Fig. 1, are given by open circles, collisions resulting in the wide structures of Fig. 2 are given by crosses in circles, and merging solitons as in Fig. 3 by crosses. Three regions are visible, labeled 1, 2, and 3, and tentative boundaries are indicated by broken lines. For large parameters K ($K>1.8$) solitons with $v<0.55$ merge, while the others pass through one another. For smaller K, three possibilities, depending on v, exist.

The two possibilities of regions 1 and 3 were found earlier, and a boundary between the two was given[8] as $S=0$ (solid line in Fig. 7), in reasonable but not perfect

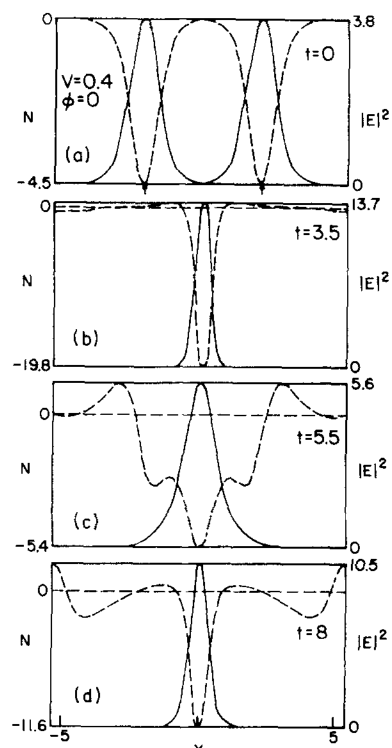

FIG. 3. The electrostatic energy density $|E|^2$ (solid line), and the ion density perturbation N (broken line) for a collision between two equal solitons with opposite velocities. The scale on the right is for $|E|^2$, on the left for N. The strength parameter $K=1.5$, the velocity $v=0.4$, and the phase difference $\varphi=0$. Note that the scale is different from the previous figures.

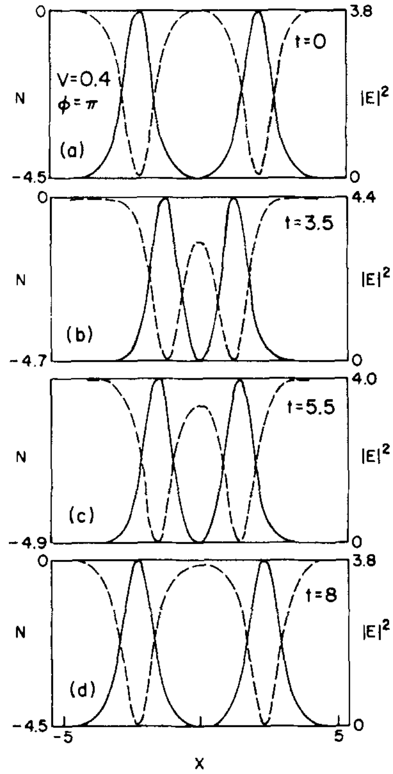

FIG. 4. The electrostatic energy density $|E|^2$ (solid line), and the ion density perturbation N (broken line) for a collision between two equal solitons with opposite velocities. The scale on the right is for $|E|^2$, on the left for N. The strength parameter $K=1.5$, the velocity $v=0.4$, and the phase difference $\varphi=\pi$. Note that the scale is different from the previous figures.

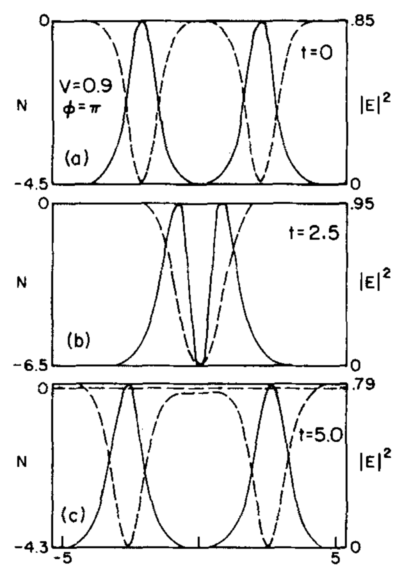

FIG. 5. The electrostatic energy density $|E|^2$ (solid line), and the ion density perturbation N (broken line) for a collision between two equal solitons with opposite velocities. The scale on the right is for $|E|^2$, on the left for N. The strength parameter $K=1.5$, the velocity $v=0.9$, and the phase difference $\varphi=\pi$. Note that the scale is different from the previous figures.

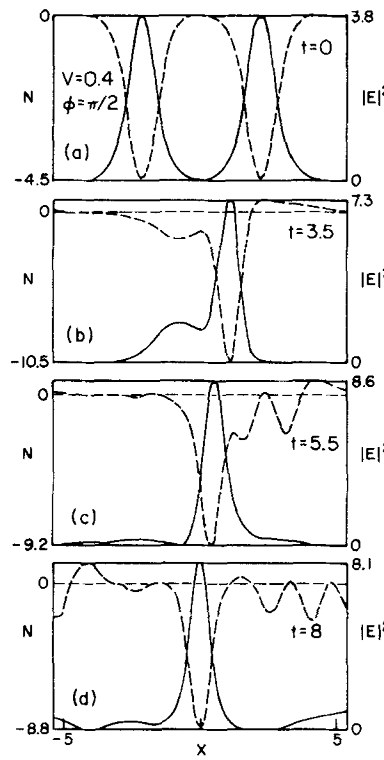

FIG. 6. The electrostatic energy density $|E|^2$ (solid line), and the ion density perturbation N (broken line) for a collision between two equal solitons with opposite velocities. The scale on the right is for $|E|^2$, on the left for N. The strength parameter $K=1.5$, the velocity $v=0.4$, and the phase difference $\varphi=\pi/2$. Note that the scale is different from the previous figures.

agreement with the boundary observed here. However, the wide structures of region 2, where K is of order unity, were not found before. Also, the importance of the relative phase difference was not recognized.

Even the simplest collision, between two equal solitons with opposite velocities, yields fairly complicated outcomes. Therefore, we have not done systematic parameter studies of collisions between different solitons, or solitons and sound waves.[2]

IV. ESTIMATES FOR COLLISION POSSIBILITIES

We now offer some estimates for the parameters separating the different outcomes of the collisions shown in

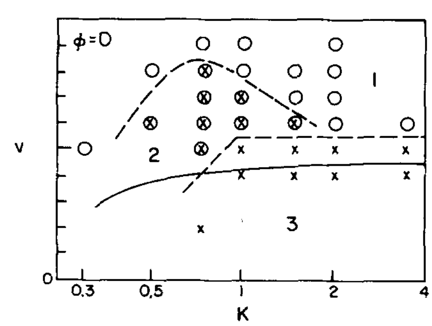

FIG. 7. Outcome of a collision as function of the parameters K and v, for $\phi=0$. Parameters in region marked 1 yield outcome as in Fig. 1, marked 2 as in Fig. 2, and 3 as in Fig. 3. Broken lines indicated approximate boundaries between regions. The solid line is $S=0$.

Figs. 1–6. We only use invariance properties and the conservation laws. Note, however, that the phase difference ϕ between the electric fields of well-separated solitons does not enter the conservation laws; thus, it is impossible to predict all collision properties for distinct ϕ's on the basis of the conservation laws alone. We only consider the simplest cases, $\phi = 0$ and $\phi = \pi$.

When $\phi = \pi$, Figs. 4 and 5, the initial condition (10) for the electric field E is antisymmetric in x, but for the ion density symmetric. The evolution equation, (1), is linear in E and symmetric in x, provided the ion density stays symmetric. Also Eq. (2) is symmetric in x, and if $|E|^2$ is also symmetric $N(x, t)$ keeps its invariance properties. Thus, the field E stays antisymmetric, and zero at $x = 0$, so that $|E|^2$ is symmetric and is zero at $x = 0$ and the fields do not overlap.

Although the energy densities stay separated, the ion densities can overlap, depending on the initial velocity. When ion density overlap occurs, the ion density behaves as in linear wave propagation, where amplitudes add on collision. We obtain the separation velocity by comparing the interaction energy S_I (Table I, No. 6, with $a = 0$) to the ion kinetic energy S_K. S_K is equal to S_I for the separation velocity $v_s^2 = 2/3$, or $v_s \simeq 0.82$, which is in reasonable agreement with the observations.

The case of zero phase angle is more complicated. We observed in Fig. 7 that, depending on the parameter K, two or three possibilities exist for the outcome of a collision. We noted in the discussion of Eq. (7) that we should expect a simple region for large K, and a more complicated region for K around one, in agreement with Fig. 7. Also, a simple region for $K \ll 1$ should be expected. This region is not of much interest, so that we have few computations there, but the tendency to fewer possibilities than around $K = 1$ is indeed observed.

Solitons that pass through one another, for large K, keep their shape approximately. We derive a boundary between regions 1 and 3 by equating the ion wave energy with *half* the interaction energy S_I. This is the energy difference between N and $|E|^2$ at the same place, and a half-width 1.33 K^{-1} apart. We find $v_s^2 = \frac{1}{3}$ or $v_s \simeq 0.58$, which agrees well with the boundary between regions 1 and 3 in Fig. 7.

Emission of ion sound waves in a collision can prevent the solitons from separating, in the following way. Assume that small amounts of ion wave energy $\int (N^2/2 + V^2/2)dx > 0$ are emitted and that the solitons keep their shape, changing their parameter K and velocity v (compare Fig. 1). The change in K on emission of a small amount of ion wave energy ΔS is $\Delta K = -\partial S/\partial K|_W \Delta S$, where

$$\left.\frac{\partial S}{\partial K}\right|_W = \frac{8K^2}{3} + \frac{40K^2v^2}{3} + (1 - v^2)^2 \qquad (11)$$

is the derivative of the energy S at constant mass W with respect to K. This derivative is positive definite, so that emission of ion sound results in wider solitons. The increase in width decreases the velocity according to $\Delta(v^2) = \partial(v^2)/\partial K|_W \Delta K$, where

$$\left.\frac{\partial(v^2)}{\partial K}\right|_W = \frac{1 - v^2}{K}, \qquad (12)$$

which follows from conservation of the invariant W.

These estimates are valid for emission of infinitesimal quantities of ion sound energy. However, as the derivatives (11) and (12) are positive definite, the conclusion that solitons decrease their velocity and increase their widths is also valid when finite amounts of ion sound energy are emitted.

As seen in Figs. 3(c) and (d) the electric field energy oscillates in amplitude and width. The oscillation can be considered as a driving term in the wave equation for the ion density, neglecting the effect of the ion density back on the oscillation for the moment. As always, such an oscillating driving term emits waves for as long as it oscillates. This process accounts for the decay in the oscillation amplitude.[8] Collided solitons will continue to emit ion sound until they have reached a stationary state, which is, of course, a soliton with $v = 0$, and $K_R = \sqrt{2}K$ determined from conservation of electrostatic energy W.

The maximum amount of ion wave energy ΔS_t that can be emitted from solitons colliding with velocity v is

$$\Delta S_t = 4v^2(1 - v^2)K(5K^2/3 + 1).$$

The maximum sound energy emitted occurs for solitons with $v^2 = \frac{1}{3}$

$$\Delta S_t < \frac{40}{27}K^3,$$

slightly more than S for a soliton with $v = 0$.

In conclusion, it is clear from this and the previous section that collisions in Eqs. (1) and (2) show quite complicated behavior, which is also different from the collisions in Eq. (5).

After submission of this article very similar work[10] came to my attention.

Note added in proof. References 11 and 12 are also relevant to the problem discussed here.

ACKNOWLEDGMENTS

Stimulating discussions with Dr. R. N. Sudan, and interest by Dr. J. Denavit are gratefully acknowledged.

This research was supported by the U. S. Energy Research and Development Administration under Contract E(11-1)3170.

*Present address: Lawrence Berkeley Laboratory, Berkeley, Calif. 94720.

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